If it's just turning the crank it's algebra, but if it's got an idea in it, then it's topology.

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Read §4.4, 4.5, 5.1, 5.4, 5.5 in Dummit and Foote.

Exercise 1. For $n \geq 3$, show $\mathrm{Aut}(D_n) \cong \mathrm{Aff}(\mathbb{Z}/(n))$. (Hint: Writing $D_n = \langle r, s\rangle$ as usual, decide where an automorphism of D_n could *possibly* send the generators r and s, and then show all those possibilities truly work, which will lead to a parametrization $\varphi_{a,b}$ of the automorphisms of D_n by parameters $a \in (\mathbb{Z}/(n))^\times$ and $b \in \mathbb{Z}/(n)$.)

Exercise 2. Let H and K be groups and $\varphi\colon K \to \mathrm{Aut}(H)$ be a homomorphism.

(a) For any automorphism $f\colon K \to K$, $\varphi \circ f$ is a homomorphism $K \to \mathrm{Aut}(H)$. Show the groups $H \rtimes_\varphi K$ and $H \rtimes_{\varphi\circ f} K$ are isomorphic.

(b) For any automorphism $f\colon H \to H$, let γ_f be conjugation by f in $\mathrm{Aut}(H)$, *i.e.*, $(\gamma_f(\alpha))(h) = f(\alpha(f^{-1}(h)))$. Then $\gamma_f \circ \varphi$ is a homomorphism $K \to \mathrm{Aut}(H)$. Show the groups $H \rtimes_\varphi K$ and $H \rtimes_{\gamma_f\circ\varphi} K$ are isomorphic.

Exercise 3. Let's classify the groups of order $2013 = 3 \cdot 11 \cdot 61$.

(a) If G has order 2013, show it has unique subgroups of orders 11 and 61, and both are normal.

(b) Let P be the subgroup of order 11 and Q be the subgroup of order 61. Show the set $PQ = \{xy : x \in P, y \in Q\}$ is a subgroup and $PQ \cong P \times Q \cong \mathbb{Z}/(671)$.

(c) Prove $G \cong \mathbb{Z}/(671) \rtimes_\varphi \mathbb{Z}/(3)$ for some homomorphism $\varphi\colon \mathbb{Z}/(3) \to (\mathbb{Z}/(671))^\times$ and use a computer algebra system to count the number of possible φ (it is more than 2).

(d) Use exercise 2 to prove the nonabelian semidirect products in part c are isomorphic, so there are **two** groups of order 2013 up to isomorphism: one abelian and one nonabelian.

Exercise 4. If G_1 and G_2 are groups with normal subgroups $N_1 \lhd G_1$ and $N_2 \lhd G_2$, then $N_1 \times N_2 \lhd G_1 \times G_2$ and $(G_1 \times G_2)/(N_1 \times N_2) \cong G_1/N_1 \times G_2/N_2$. Your task is to generalize this to semidirect products. Let $H \rtimes_\varphi K$ be a semidirect product defined by some action $\varphi\colon K \to \mathrm{Aut}(H)$, so $(h, k)(h', k') = (h\varphi_k(h'), kk')$. We write $H \rtimes_\varphi K$ as $H \rtimes K$ to avoid cluttering the notation.

Given normal subgroups $N \lhd H$ and $M \lhd K$, we'd like to know if

$$(H \rtimes K)/(N \rtimes M) \cong H/N \rtimes K/M,$$

where the actions of M on N (to define $N \rtimes M$) and K/M on H/N (to define $H/N \rtimes K/M$) should come in a reasonable way from that of K on H that defines $H \rtimes K$ (namely, from φ).

For $N \rtimes M$ and $H/N \rtimes K/M$ to make sense using K's action on H, we want

- the action of K on H to preserve N (*i.e.*, $\varphi_k(N) \subset N$ for each $k \in K$), so K acts on H/N and (by restricting the domain of φ) $M \subset K$ acts on N, thus giving meaning to $N \rtimes M$.
- the action of $M \subset K$ on H/N to be trivial (that is, $\varphi_m(h) \in hN$ for all $m \in M$ and $h \in H$), so the action of K on H/N induces an action of K/M on H/N.

Now for the exercise: under the two conditions above, show $N \rtimes M \lhd H \rtimes K$ and there is an isomorphism as displayed above. (For example, using $M = K$, $(H \rtimes K)/(N \rtimes K) \cong H/N$ if $N \lhd H$ and $\varphi_k(h) \in hN$ for all $k \in K$ and $h \in H$.)

Remark. If we assume only the first condition, then $N \rtimes M \lhd H \rtimes K$ if and only if the action of M on H/N is trivial, but you are not asked to show this.