I have often pondered over the roles of knowledge or experience, on the one hand, and imagination or intuition, on the other, in the process of discovery. I believe that there is a certain fundamental conflict between the two, and knowledge, by advocating caution, tends to inhibit the flight of imagination. Therefore, a certain naivete, unburdened by conventional wisdom, can sometimes be a positive asset.

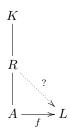
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Read: text sections 7.1-7.4, 8.1-8.2

Exercise 1. Let R be a (nonzero) commutative ring. An element $x \in R$ is called *nilpotent* if $x^n = 0$ for some n.

- (a) Find all the nilpotent elements of $\mathbb{Z}/(9)$ and $\mathbb{Z}/(27)$.
- (b) Describe (with justification) all the nilpotent elements of $\mathbb{Z}/(p^k)$ where p is prime and $k \geq 2$.
- (c) Describe (with justification) all the nilpotent elements of $\mathbb{Z}/(20)$ and then all the nilpotent elements of any $\mathbb{Z}/(m)$ for $m \geq 2$. (Note: these are generally *not* all the non-invertible elements!)
- (d) Show the nilpotent elements of R form an ideal. (Hint for addition: if $x^n = 0$ and $y^m = 0$, show $(x + y)^{n+m} = 0$ by thinking about what powers of x and y can show up when expanding this power.)
- (e) If x is nilpotent, show 1 + x is invertible. (Hint: geometric series!)

Exercise 2. Let K and L be fields and A be a subring of K that admits a homomorphism $f: A \to L$. We want to extend f to a homomorphism on larger subrings of K (see diagram below). We can't always extend f all the way to K, e.g., the reduction map $\mathbb{Z} \to \mathbb{Z}/(p)$, for prime p can't be extended to a homomorphism $\mathbb{Q} \to \mathbb{Z}/(p)$.



- (a) If $a \in A \{0\}$ and $f(a) \neq 0$, show f admits a unique extension to a ring homomorphism $A[1/a] \to L$. (It is insufficient to show there is at most one extension of f to A[1/a]: you must show there really is an extension of f to A[1/a].)
- (b) Use Zorn's lemma to show there is an extension of f to a ring homomorphism $\widetilde{f} \colon \widetilde{A} \to L$ for some ring \widetilde{A} between A and K, and \widehat{f} can't be extended further to a ring homomorphism $R \to L$ for any R strictly between \widetilde{A} and K.
- (c) If $x \in \widetilde{A}$ and $\widetilde{f}(x) \neq 0$, use parts a and b to show $1/x \in \widetilde{A}$. Conclude that $\ker \widetilde{f}$ is a maximal ideal in \widetilde{A} , and in fact is the only maximal ideal in \widetilde{A} .

Exercise 3. Let d be a nonsquare integer. For $\alpha = a + b\sqrt{d}$ in $\mathbb{Z}[\sqrt{d}]$, where a and b are integers, we set $\overline{\alpha} = a - b\sqrt{d}$ and $N(\alpha) = \alpha \overline{\alpha} = a^2 - db^2$. Then $N(\alpha) \in \mathbb{Z}$, and $N(\alpha) \neq 0$ if $\alpha \neq 0$. The task of this problem is to give a combinatorial interpretation for $N(\alpha)$, or rather for $|N(\alpha)|$.

(a) For nonzero integers m, show $\mathbb{Z}[\sqrt{d}]/(m)$ has size m^2 .

- (b) For nonzero α , show multiplication by $\overline{\alpha}$ is an additive group isomorphism $\mathbb{Z}[\sqrt{d}]/(\alpha) \to (\overline{\alpha})/(N(\alpha))$ and conjugation induces an additive group isomorphism $(\alpha)/(N(\alpha)) \to (\overline{\alpha})/(N(\alpha))$.
- (c) Conclude from parts a and b that the ring $\mathbb{Z}[\sqrt{d}]/(\alpha)$ has size $|N(\alpha)|$. (Hint: Think of $\mathbb{Z}[\sqrt{d}]/(\alpha)$ as an additive group.)
- **Exercise 4.** (a) Mimic the proof that $\mathbb{Z}[i]$ is Euclidean to show $\mathbb{Z}[\sqrt{-2}]$ is Euclidean with respect to the function $N(a+b\sqrt{-2})=a^2+2b^2$. That is, for any α and β in $\mathbb{Z}[\sqrt{-2}]$ with $\beta \neq 0$ show there are γ and ρ in $\mathbb{Z}[\sqrt{-2}]$ such that $\alpha = \beta \gamma + \rho$ and $N(\rho) < N(\beta)$.
- (b) Mimic the proof that $\mathbb{Z}[i]$ is Euclidean to show $\mathbb{Z}[\sqrt{2}]$ is Euclidean with respect to the function $|N(a+b\sqrt{2})|=|a^2-2b^2|$. That is, for any α and β in $\mathbb{Z}[\sqrt{2}]$ with $\beta\neq 0$ show there are γ and ρ in $\mathbb{Z}[\sqrt{2}]$ such that $\alpha=\beta\gamma+\rho$ and $|N(\rho)|<|N(\beta)|$. (Warning: don't make mistakes with inequalities, like saying $|x-y|\leq |x|-|y|!$)
- (c) Use your method of proof in parts and b to solve $8+20\sqrt{-2}=(5+4\sqrt{-2})\gamma+\rho$ in $\mathbb{Z}[\sqrt{-2}]$ with $N(\rho)< N(\beta)$ and $27+7\sqrt{2}=(7+3\sqrt{2})\gamma+\rho$ in $\mathbb{Z}[\sqrt{2}]$ with $|N(\rho)|<|N(\beta)|$.

Exercise 5. Decide which of the following ideals in $\mathbb{Z}[X]$ are prime or maximal by identifying the quotient rings by these ideals with more familiar rings:

- (a) (3)
- (b) (10)
- (c) (5, X)
- (d) $(X^2 3)$
- (e) $\{f(X) \in \mathbb{Z}[X] : f(2) = 0\}.$