We may as well cut out group theory [from the curriculum]. That is a subject which will never be of any use in physics. J. Jeans (1910)

Read §1.7, 4.1– 4.4 (Dummit and Foote).

Exercise 1. For a subgroup H of the group G, its normalizer is $\mathrm{N}_G(H) = \{g \in G : gHg^{-1} = H\}$.

(a) In the group $\mathrm{GL}_2(\mathbb{R})$ show the normalizer of the subgroup $\mathrm{Aff}(\mathbb{R}) = \{\left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) : x \neq 0\}$ is $\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a, d \neq 0\}$.

(b) In the group $\mathrm{GL}_2(\mathbb{R})$ show the normalizer of the diagonal subgroup $D = \{\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) : x, y \neq 0\}$ is $\{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) : a, b, c, d \neq 0\}$.

(c) View S_n as $\mathrm{Sym}(\mathbb{Z}/(n))$ instead of as $\mathrm{Sym}(\{1, 2, \ldots, n\})$. If a and b are integers such that $(a, n) = 1$, let $\sigma_{a,b} \colon \mathbb{Z}/(n) \to \mathbb{Z}/(n)$ by $\sigma_{a,b}(x \bmod n) = ax + b \bmod n$. Show $\sigma_{a,b}$ belongs to the normalizer in S_n of the subgroup $\langle (123 \cdots n) \rangle$ and, conversely, any element of the normalizer of $\langle (123 \cdots n) \rangle$ in S_n is some $\sigma_{a,b}$. Conclude that $\mathrm{N}_{S_n}(\langle 123 \cdots n \rangle) \cong \mathrm{Aff}(\mathbb{Z}/(n))$.

Solution 1.

Exercise 2. (Related to exercise 11, §4.3) For the following permutations σ_1 and σ_2, compute a permutation π such that $\pi\sigma_1\pi^{-1} = \sigma_2$.

(a) $\sigma_1 = (12)(345)$, $\sigma_2 = (123)(45)$

(b) $\sigma_1 = (12)(34)(56)$, $\sigma_2 = (35)(24)(16)$

(c) $\sigma_1 = (15)(372)(4689)$, $\sigma_2 = \sigma_1^{-1}$

Solution 2.

Exercise 3. Let $\mathrm{SL}_2(\mathbb{Z})$ act on nonzero vectors in $\mathbb{Z}^2$ by multiplication: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\binom{x}{y} = \binom{ax+by}{cx+dy}$.

(a) When m and n are distinct positive integers, show $\binom{m}{0}$ and $\binom{n}{0}$ lie in distinct orbits.

(b) For any nonzero vector $\binom{x}{y}$ in $\mathbb{Z}^2$, show its $\mathrm{SL}_2(\mathbb{Z})$-orbit contains $\binom{m}{0}$, where $m = \gcd(x, y)$. (Thus the orbits for this action are the vectors in $\mathbb{Z}^2$ with the same greatest common divisor.)

Solution 3.

Exercise 4. (Related to exercise 10, §4.1) Let G be a group and fix subgroups H and K of G. We define an HK *double coset* to be a subset of G having the form $HgK = \{hgk : h \in H, k \in K\}$, where $g \in G$. This construction is usually not symmetric in the roles of H and K.

(a) Show the rule
$$(h, k)g := hgk^{-1}$$
defines a group action of $H \times K$ on the set G, and that the HK double cosets are the orbits for this action. (In particular, it then follows from the theory of group actions that different HK double cosets are disjoint.)

(b) Why does the rule "$(h, k)x := hxk$" *not* generally define an action of $H \times K$ on G?

(c) Compute all double cosets HgK (*no* repetitions) when $G = S_3$, $H = \langle (12) \rangle$, and $K = \langle (13) \rangle$, and all the distinct double cosets $H + g + K$ in $G = \mathbb{Z}/(30)$ (additive cyclic group) where $H = \langle 6 \rangle$, and $K = \langle 15 \rangle$. In the first case there are orbits whose size is not a factor of $\#S_3 = 6$. Why does this not contradict the fact that the size of an orbit divides the size of the group?

(d) Using left multiplication, H acts on any double coset HgK. Prove each orbit for this action of H on HgK is a right coset of H in G (not just a subset of a right coset in G). In particular, when G is finite, conclude that $\#H$ divides $\#(HgK)$. What group action lets you conclude that $\#K$ divides $\#(HgK)$?

(e) Show the number of H-orbits in HgK for the action in part d is $[K : K \cap g^{-1}Hg]$. Conclude that $\#(HgK) = (\#H)[K : K \cap g^{-1}Hg]$.

Solution 4.