The main difficulty for the beginner is to absorb a reasonable vocabulary in a short time. None of the concepts is difficult, but there is an accumulation of new concepts which may sometimes seem heavy. S. Lang

Read §1.2–1.6, 2.4, 3.1–3.3, 3.5 (Dummit and Foote).

Exercise 1. Let G be a cyclic group, and let H be an arbitrary subgroup. Prove that G is abelian, and that H is abelian also. Moreover, if G is finite, and H' is another subgroup $H' \subseteq G$ such that $|H| = |H'|$, then $H = H'$. (Here $|H|$ denotes the order, the size, the number of elements of H.)

Solution 1.

Exercise 2. (a) For complex numbers z and w with $(z, w) \neq (0, 0)$, set

$$f(z, w) := \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}).$$

Fill in the following equations, and indicate why the numbers you use are not both 0:

$$f(z, w)f(u, v) = f(?, ?), \quad f(z, w)^{-1} = f(?, ?).$$

This shows the set of matrices $f(z, w)$ with $(z, w) \neq (0, 0)$ is a subgroup of $\mathrm{GL}_2(\mathbb{C})$. (Note: The eight matrices where one of z or w is ± 1 or $\pm i$ and the other is 0 is a concrete model of the quaternion group Q_8.)

(b) On the set $\mathbb{C}^2 - \{(0, 0)\}$ consider the following binary operation:

$$(z, w)(u, v) = (zu + iw\overline{v}, zv + \overline{u}w).$$

(This operation on $\mathbb{C}^2 - \{(0, 0)\}$ nearly resembles the multiplication in part a.) Show there is a 2-sided identity and each pair in $\mathbb{C}^2 - \{(0, 0)\}$ has a left inverse and a right inverse, but that these inverses are usually *not* the same and that the operation is *not* associative.

Solution 2.

Exercise 3. Express each of the permutations (15)(1234)(12563) and (123)(246)(345) in S_6 as a product of disjoint cycles and as a product of transpositions. Then determine their order.

Solution 3.

Exercise 4. Fix a group G and a positive integer m. Let P_m be the subgroup of G that is generated by the set of mth powers $\{x^m : x \in G\}$.

(a) Show $P_m \lhd G$ and $\overline{g}^m = 1$ in the quotient group G/P_m for all $g \in G$.

(b) For any group homomorphism $f \colon G \to H$ such that $f(g)^m = 1$ for all $g \in G$, show there is a unique group homomorphism $f_m \colon G/P_m \to H$ such that $f_m(\overline{g}) = f(g)$ for all $g \in G$.

(c) Show the commutator subgroup of G lies inside P_2. (Hint: show G/P_2 is abelian.)

(d) If $G = D_n$, show $P_m = D_n$ when m is odd and $P_m = \langle r^m \rangle$ when m is even.

(e) For any finite group G show $P_m = P_{(m,|G|)}$.

Solution 4.

Exercise 5. Let $f \colon G_1 \to G_2$ be a group isomorphism.

(a) If $g \in G_1$, show g and $f(g)$ have the same order (allow for the possibility of g or $f(g)$ having infinite order).

(b) Show f induces isomorphisms between the centers of G_1 and G_2, and between the commutator subgroups of G_1 and G_2.

(c) If H is a subgroup of G_1, show f induces a bijection between the left cosets of H in G_1 and the left cosets of $f(H)$ in G_2, so $[G_1 : H] = [G_2 : f(H)]$ as cardinal numbers.

(d) If N is a normal subgroup of G_1, show $f(N)$ is a normal subgroup of G_2 and f induces a group isomorphism between G_1/N and $G_2/f(N)$.

Solution 5.