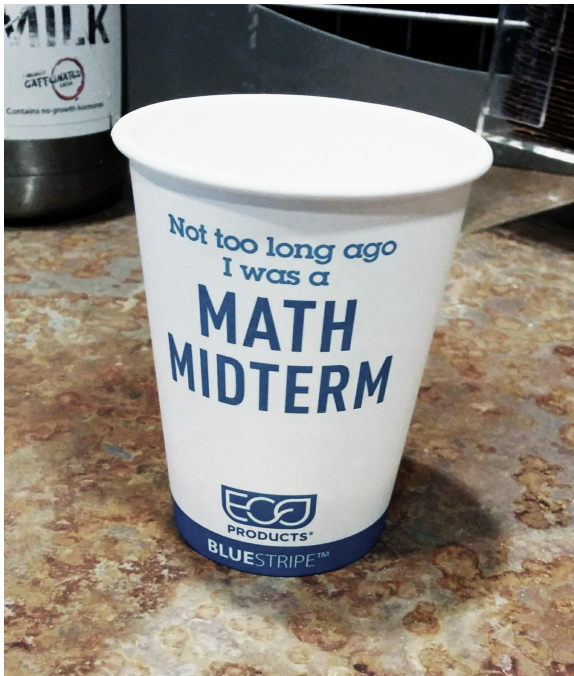




Not too long ago
I was a
**MATH
MIDTERM**



PRODUCTS®
BLUESTRIPE™

Our first Midterm Exam is TODAY!

- Two times (check your time following these instructions)
 - (1) 6-8 PM at AUST 108, or
 - (2) 9-11 PM at TLS 154.
- There is a practice exam and solutions in outline in the website:
<http://alozano.clas.uconn.edu/math1131f14>
- Covers Sections 1.1 - 3.3 (including 3.3, derivatives of trigonometric functions; does not include chain rule).
- Today's class: review.
- There was a review session on Monday 9/29, now available here.

MATH 1131Q - Calculus 1.

Álvaro Lozano-Robledo

Department of Mathematics
University of Connecticut

Day 11 - Review day

Midterm 1:

Functions

Limits

Continuity

Derivatives

Clicker Question: I struggle the most with...

- (A) Finding inverse functions.
- (B) Finding asymptotes.
- (C) The ε - δ definition of limit.
- (D) The limit definition of derivative.
- (E) Product and quotient rules of differentiation.

Functions

Definition

A **function** is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E .

- The set D is called the **domain** of the function f .
- The **range** of f is the set of all possible values $f(x)$ as x varies throughout the elements of D .

Definition

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function (with domain $\mathbb{R}$ and values in $\mathbb{R}$). The **graph** of f is the set $\{(x, y) : y = f(x), \text{ for } x \in \mathbb{R}\} = \{(x, f(x)) : x \in \mathbb{R}\}$.

Definition

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function.

- 1 We say that f is **increasing** if $f(a) < f(b)$ whenever $a < b$, and
- 2 We say that f is **decreasing** if $f(a) > f(b)$ whenever $a < b$.

Catalog of elementary functions

- 1 Linear functions, $f(x) = mx + b$, for some constants m, b .
- 2 Power functions, $f(x) = x^a$, for some real number a .
- 3 Polynomials, $f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$, for some constants a_i , and non-negative integer n .
- 4 Rational functions $f(x) = \frac{p(x)}{q(x)}$, where p and $q \neq 0$ are polynomials.
- 5 Exponential functions, $f(x) = a^x$, where $a > 0$.
- 6 Trigonometric functions, $f(x) = \sin(x), \cos(x)$.

New functions from old functions

We can create new functions from elementary functions using the following transformations. Let $f(x)$ be a fixed function, and we define a new function $g(x)$ as follows:

- 1 **Vertical shift** of C units: $g(x) = f(x) + C$, where C is a constant.
- 2 **Horizontal shift** of C units: $g(x) = f(x - C)$.
- 3 **Vertical stretch**: $g(x) = Cf(x)$, with $C > 0$.
- 4 **Horizontal stretch**: $g(x) = f(Cx)$, with $C > 0$.
- 5 **Vertical reflection** about x -axis: $g(x) = -f(x)$.
- 6 **Horizontal reflection** about y -axis: $g(x) = f(-x)$.

Let $f(x)$, $g(x)$ be fixed functions, and we define a new function $h(x)$ as follows:

- 1 **Add/subtract functions**, $h(x) = f(x) + g(x)$.
- 2 **Multiply/divide functions**, $h(x) = f(x)g(x)$, $h(x) = f(x)/g(x)$, where $g(x) \neq 0$.
- 3 **Composition of functions**, $h(x) = f(g(x))$.

Inverse Functions

Definition

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. We say f is **one-to-one** (or 1-1, or injective), if it never takes the same value twice, i.e.,

$$f(a) \neq f(b) \text{ whenever } a \neq b,$$

or, equivalently,

$$\text{if } f(a) = f(b), \text{ then } a = b.$$

Definition

Let f be a one-to-one function with domain A and range B . Then, its inverse function f^{-1} has domain B and range A , and is defined by

$$f^{-1}(y) = x \iff f(x) = y,$$

for any y in B .

Definition

The inverse function f^{-1} is defined by

$$f^{-1}(y) = x \iff f(x) = y.$$

Example (Practice Exam, Problem 2(a))

If $f(x) = x/(x + 3)$ for $x \neq -3$, then find its inverse function $f^{-1}(x)$ and determine the domain of the inverse function.

Definition

The inverse function f^{-1} is defined by

$$f^{-1}(y) = x \iff f(x) = y.$$

Example (Practice Exam, Problem 2(a))

If $f(x) = x/(x+3)$ for $x \neq -3$, then find its inverse function $f^{-1}(x)$ and determine the domain of the inverse function.

(A) $f^{-1}(x) = (x-3)/x$

(B) $f^{-1}(x) = 3x/(1-x)$

(C) $f^{-1}(x) = x/(3x-1)$

(D) $f^{-1}(x) = x(x+3)$

(E) I really could use some help with this one.

$$y = \frac{x}{x+3} \implies x = \frac{y}{y+3}$$

Solve for y : $x(y+3) = y$

$$xy + 3x = y$$

$$(x-1)y = -3x$$

$$y = \frac{-3x}{x-1} = \frac{3x}{1-x}$$

Definition

The inverse function f^{-1} is defined by

$$f^{-1}(y) = x \iff f(x) = y.$$

Example (Practice Exam, Problem 2(a))

If $f(x) = x/(x + 3)$ for $x \neq -3$, then find its inverse function $f^{-1}(x)$ and determine the domain of the inverse function.

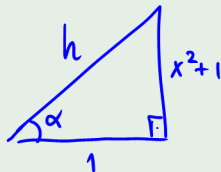
Definition

The inverse function f^{-1} is defined by

$$f^{-1}(y) = x \iff f(x) = y.$$

Example

Simplify the following expression: $\cos(\underbrace{\tan^{-1}(x^2 + 1))}_{\alpha})$.



$$\tan \alpha = \frac{x^2 + 1}{1} = x^2 + 1$$

by Pythagoras: $h = \sqrt{(x^2 + 1)^2 + 1^2} = \sqrt{x^4 + 2x^2 + 2}$

$$\cos(\tan^{-1}(x^2 + 1)) = \cos(\alpha) = \frac{1}{h}$$

$$= \frac{1}{\sqrt{x^4 + 2x^2 + 2}}$$

another similar problem
 $\sin(\cos^{-1}(4x))$

Limits

Definition (Informal Definition of Limit)

Let $f(x)$ be a function that is defined when x is near the number a (i.e., f is defined on some open interval that contains a , except possibly a itself). Then, we write

$$\lim_{x \rightarrow a} f(x) = L$$

and we say the limit of $f(x)$, as x approaches a , equals L , if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a .

Also:

- Sided limits: $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$,
- Infinite limits: $\lim_{x \rightarrow a} f(x) = \infty$, and
- Limits at infinity: $\lim_{x \rightarrow \infty} f(x) = L$.

The Limit Laws

If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

- $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} (cf(x))$, where c is a constant, $= c \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} (f(x)g(x)) = \left(\lim_{x \rightarrow a} f(x) \right) \left(\lim_{x \rightarrow a} g(x) \right)$
- $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$, if $\lim_{x \rightarrow a} g(x) \neq 0$.
- $\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n$
- $\lim_{x \rightarrow a} (f(x))^{n/m} = \left(\lim_{x \rightarrow a} f(x) \right)^{n/m}$, provided $f(x) \geq 0$ for x near a when m is even.

More Limit Laws

- $\lim_{x \rightarrow a} C = C$
- $\lim_{x \rightarrow a} x = a$
- $\lim_{x \rightarrow a} x^n = a^n$, where n is a positive integer.
- $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer (If n is even we assume that $a > 0$)
- $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$, where n is a positive integer (If n is even we assume that $\lim_{x \rightarrow a} f(x) > 0$)

Example

Determine the following limit, **using algebraic methods** to simplify the expression before finding the limit.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{\sqrt{x}}$$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{x+4} - 2)(\sqrt{x+4} + 2)}{\sqrt{x}(\sqrt{x+4} + 2)} = \lim_{x \rightarrow 0} \frac{x+4 - 4}{\sqrt{x}(\sqrt{x+4} + 2)}$$

$$= \lim_{x \rightarrow 0} \frac{x}{\sqrt{x}(\sqrt{x+4} + 2)} = \lim_{x \rightarrow 0} \frac{\sqrt{x}}{(\sqrt{x+4} + 2)} = \frac{0}{4} = 0.$$

$$\rightarrow \frac{x}{\sqrt{x}} \dots = \frac{x}{x^{1/2}} \dots = \frac{x^{1-1/2}}{\dots} = \frac{x^{1/2}}{\dots} = \frac{\sqrt{x}}{\dots}$$

Example

Determine the following limit, **using algebraic methods** to simplify the expression before finding the limit.

$$\lim_{x \rightarrow 0^+} \frac{\sqrt{x+4} - 2}{\sqrt{x}} = 0$$

Not defined for $x \leq 0$!!



Example (Practice Exam, Problem 4(a))

Determine the following limit, **using algebraic methods** to simplify the expression before finding the limit.

$$\lim_{x \rightarrow 2} \frac{3-x}{1/3 - 1/x}$$

- This is a continuous function in its domain
- $x=2$ is in the domain $\mathbb{R} - \{0, 3\}$

$$\Rightarrow \text{evaluate } \underline{\text{limit}} = \frac{3-2}{1/3 - 1/2} = \frac{1}{1/3 - 1/2} = \frac{6}{2-3} = \frac{6}{-1} = -6$$

Example (Practice Exam, Problem 4(c))

Determine the following limit, **using algebraic methods** to simplify the expression before finding the limit.

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$$

$$= \lim_{x \rightarrow 4} \frac{x - 4}{(\sqrt{x} - 2)(\sqrt{x} + 2)} =$$

$$= \lim_{x \rightarrow 4} \frac{\cancel{(x - 4)}(\sqrt{x} + 2)}{\cancel{(x - 4)}} = \lim_{x \rightarrow 4} \sqrt{x} + 2 = 4$$

continuous
at $x = 4$

The FORMAL Definition of Limit

Definition (Formal Definition of Limit)

Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then, we say that the limit of $f(x)$ as x approaches a is L , and we write

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

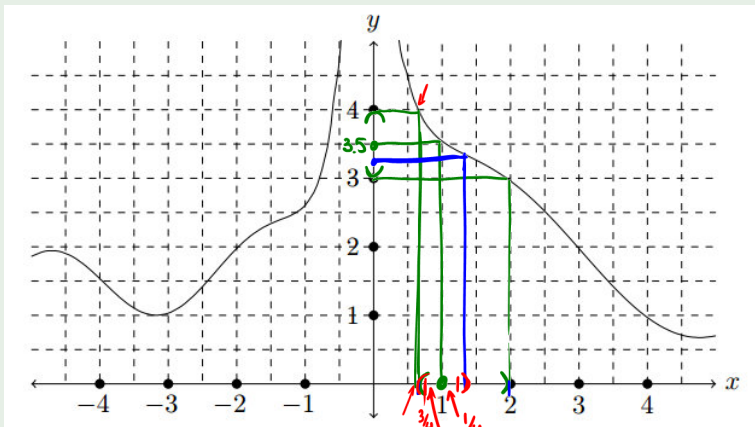
if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

radius about L

radius about a

Example (Practice Test, Problem 7)

For the function $f(x)$ graphed below, continuous except at $x = 0$, we have $f(1) = 3.5$ and $f(-2) = 2$.



(a) Determine a value of $\delta_1 > 0$ such that

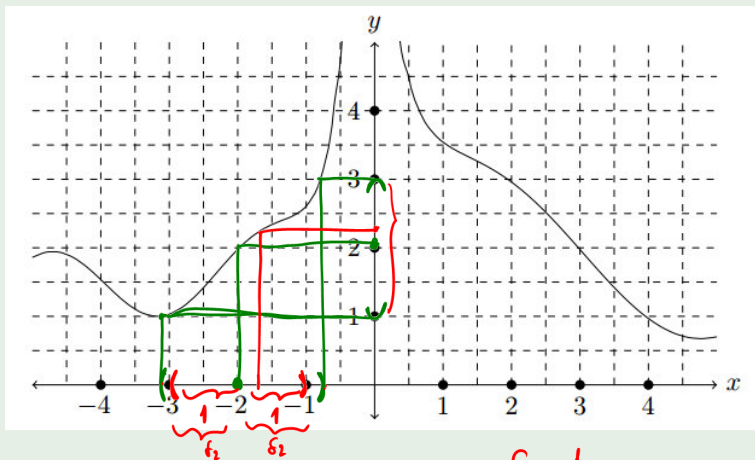
$0 < |x - 1| < \delta_1 \Rightarrow |f(x) - 3.5| < 1/2$. Your answer should be

supported by **what you draw** in the figure.

$\delta_1 = \frac{1}{4}$ would work.

Example (Practice Test, Problem 7)

For the function $f(x)$ graphed below, continuous except at $x = 0$, we have $f(1) = 3.5$ and $f(-2) = 2$.



(b) Determine a value of $\delta_2 > 0$ such that

$0 < |x - (-2)| < \delta_2 \Rightarrow |f(x) - 2| < 1$. Your answer should be

supported by **what you draw** in the figure.

$\delta_2 = 1$ works.

Theorem

Let $r > 0$ be a positive real number. Then:

$$① \quad \lim_{x \rightarrow \infty} \frac{1}{x^r} = 0,$$

$$② \quad \lim_{x \rightarrow \infty} e^x = \infty,$$

$$③ \quad \lim_{x \rightarrow 0} \log(x) = -\infty.$$

Useful trick: Suppose $p(x)$ and $q(x)$ are two polynomials. In order to calculate

$$\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)}$$

we divide first the numerator and denominator by the highest power of x that appears in the denominator.

Example (Practice Test, Problem 5)

Let $f(x) = \frac{2x^3 + x^2 - 7}{\sqrt{49x^6 + 9x^2 + x}}$. (Already discussed at the end of Day 9.)

① Compute $\lim_{x \rightarrow \infty} f(x)$. $= \frac{2}{7}$ (Divide by $x^3 = \sqrt{x^6}$ everywhere ... by x^6 inside the sqrt)

② Compute $\lim_{x \rightarrow -\infty} f(x)$. $= -\frac{2}{7}$

(Divide by $\sqrt{x^6}$ everywhere ... $\sqrt{x^6} = |x|^3$...)

$\left(\frac{x}{|x|} = -1 \text{ when } x < 0 \right)$

$\downarrow$
b/c $\sqrt{x^2} = |x|$
and not just x .

Horizontal Asymptotes

Definition (Formal Definition of Limit at Infinity)

Let f be a function defined on some open interval (a, ∞) . Then, we say that

- 1 the limit of $f(x)$ as x approaches ∞ is L , and we write $\lim_{x \rightarrow \infty} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.

(In this case, we say that the line $y = L$ is a horizontal asymptote of $f(x)$ at ∞ .)

- 2 the limit of $f(x)$ as x approaches $-\infty$ is L , and we write $\lim_{x \rightarrow -\infty} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $M > 0$ such that if $x < -M$, then $|f(x) - L| < \varepsilon$.

(In this case, we say that the line $y = L$ is a horizontal asymptote of $f(x)$ at $-\infty$.)

Example (Practice Test, Problem 1(b))

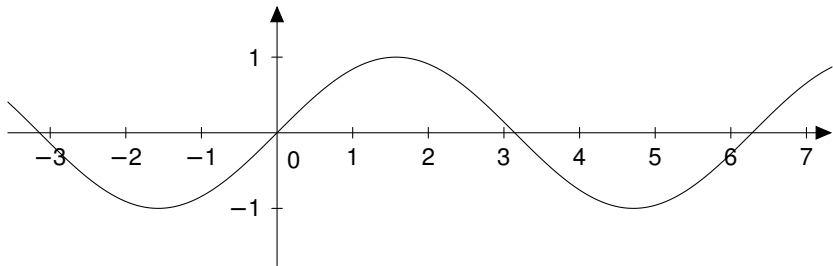
True or False (plus justification): The function $y(t) = 3 + 6e^{-kt}$, with k a positive constant, has a horizontal asymptote $y = 6$.

To find out $\lim_{t \rightarrow \infty} y(t)$ and $\lim_{t \rightarrow -\infty} y(t)$

$$\bullet \lim_{t \rightarrow \infty} 3 + 6e^{-kt} = \lim_{t \rightarrow \infty} 3 + \frac{6}{e^{kt}} = 3 \rightarrow \text{horizontal asymptote } \boxed{y=3}$$

$$\bullet \lim_{t \rightarrow -\infty} 3 + 6e^{-kt} = \lim_{\substack{\uparrow \\ -t=s}} 3 + 6e^s = \infty \quad \begin{array}{l} \text{No Horiz.} \\ \text{ASYMP.} \\ \text{AT } -\infty \end{array}$$

Continuity



The Formal Definition of Continuity

Definition

A function $f(x)$ is continuous at a number a if

- 1 $f(x)$ is defined at $x = a$, i.e., $f(a)$ is well-defined,
- 2 $\lim_{x \rightarrow a} f(x)$ exists, and
- 3 $\lim_{x \rightarrow a} f(x) = f(a)$.

Example (Practice Exam, Problem 6)

$$\text{Let } f(x) = \begin{cases} 3 - kx, & \text{if } x > 1, \\ k - x, & \text{if } x \leq 1. \end{cases}$$

$$\begin{aligned} \textcircled{1} \quad \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} k - x = k - 1 \\ \textcircled{2} \quad \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} 3 - kx = 3 - k \end{aligned} \quad \left. \vphantom{\begin{aligned} \textcircled{1} \quad \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} k - x = k - 1 \\ \textcircled{2} \quad \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} 3 - kx = 3 - k \end{aligned}} \right\} \begin{array}{l} \text{have to coincide if } f \\ \text{is continuous at } x=1 \end{array}$$

- ③ Find the constant k so that $f(x)$ is continuous at every point.

if $x > 1$, then $f(x) = 3 - kx \Rightarrow$ continuous polynomial.

if $x < 1$, then $f(x) = k - x \Rightarrow$ " "

$$\text{At } x=1, f(1) = k - 1 = 3 - k \Rightarrow 2k = 4 \Rightarrow k = 2$$

Theorems about Continuous Functions

Theorem

The following types of functions are continuous at every number in their domain: polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions.

Theorem

If $f(x)$ and $g(x)$ are continuous functions at $x = a$, and c is a constant, then the following functions are also continuous at a :

$$f(x) + g(x), f(x) - g(x), cf(x), f(x)g(x), \frac{f(x)}{g(x)} \text{ if } g(a) \neq 0.$$

Theorems about Continuous Functions

Theorem

If f is continuous at $x = b$, and $\lim_{x \rightarrow a} g(x) = b$, then

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$$

Theorem

If $g(x)$ is continuous at $x = a$, and $f(x)$ is continuous at $g(a)$, then the composite function $f(g(x))$ is continuous at $x = a$.

Theorem (The Intermediate Value Theorem)

Suppose that $f(x)$ is continuous on the interval $[a, b]$, such that $f(a) \neq f(b)$, and let R be any real number between $f(a)$ and $f(b)$. Then, there is a number c in (a, b) such that $f(c) = R$.

Derivatives

(Rates of Change)

Let $f(x)$ and $g(x)$ be differentiable functions (i.e., $f'(x)$ and $g'(x)$ exist at every point x).

Derivatives

- $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.
- $(c)' = 0$, where c is a constant.
- $(x^n)' = nx^{n-1}$, where n is real.
- $(a^x)' = ka^x$, where $a > 0$,
 $k = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$.
- $(e^x)' = e^x$.
- $(\sin x)' = \cos x$.
- $(\cos x)' = -\sin x$.

Rules

- $(cf)' = cf'$, where c is a constant.
- $(f + g)' = f' + g'$.
- Product rule: $(fg)' = f'g + fg'$.
- Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Let $f(x)$ and $g(x)$ be differentiable functions (i.e., $f'(x)$ and $g'(x)$ exist at every point x).

Derivatives

- $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.
- $(c)' = 0$, where c is a constant.
- $(x^n)' = nx^{n-1}$, where n is real.
- $(a^x)' = ka^x$, where $a > 0$,
 $k = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$.
- $(e^x)' = e^x$.
- $(\sin x)' = \cos x$.
- $(\cos x)' = -\sin x$.

Rules

- $(cf)' = cf'$, where c is a constant.
- $(f + g)' = f' + g'$.
- Product rule: $(fg)' = f'g + fg'$.
- Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.
- Chain rule:
 $(f(g(x)))' = f'(g(x))g'(x)$.
(Not in this midterm)

Continuity and Differentiability

A very important theorem: differentiability is a *stronger* condition than continuity.

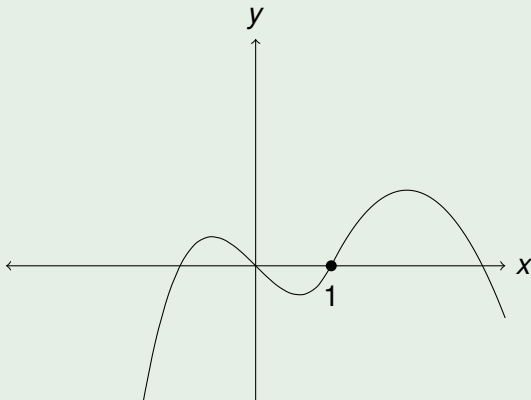
Theorem

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

Warning! The converse is not true. The function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$.

Example (Practice Test, Problem 10)

The picture below is the graph of $f(x) = \begin{cases} x^3 - x, & \text{if } x \leq 1, \\ -x^2 + ax + b, & \text{if } x > 1, \end{cases}$ where a and b are chosen to make the function differentiable at $x = 1$. Determine a and b . (Hint: the function has to be continuous at $x = 1$ also, since it is differentiable.)



Example (Practice Test, Problem 10)

The picture below is the graph of $f(x) = \begin{cases} x^3 - x, & \text{if } x \leq 1, \\ -x^2 + ax + b, & \text{if } x > 1, \end{cases}$ where a and b are chosen to make the function differentiable at $x = 1$. Determine a and b . (Hint: the function has to be continuous at $x = 1$ also, since it is differentiable.)

• It is differentiable for $x < 1$ and $x > 1$ b/c def'd by polys.

• Continuity is needed: $x^3 - x$ and $-x^2 + ax + b$ should have same value at 1.

$$1^3 - 1 = 0 = -1 + a + b \Rightarrow a + b = 1.$$

• Differentiable: Need derivatives to match.

$$f'(x) = \begin{cases} 3x^2 - 1 & \text{if } x < 1 \\ -2x + a & \text{if } x > 1 \end{cases}$$

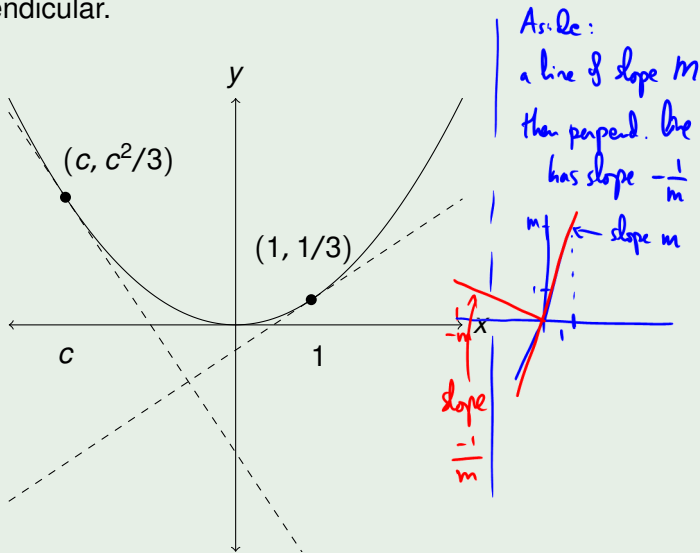
$$f'(1) = 3 \cdot 1^2 - 1 = 2 = -2 \cdot 1 + a = -2 + a$$

$$\Rightarrow a = 4$$

$$\begin{aligned} a + b &= 1 \\ 4 + b &= 1 \end{aligned} \Rightarrow \boxed{b = -3}$$

Example (Practice Test, Problem 11)

In the graph of $y = x^2/3$, which is illustrated below, find the number c such that the tangent lines to the graph at the points where $x = 1$ and $x = c$ are perpendicular.



Example (Practice Test, Problem 11)

In the graph of $y = x^2/3$, which is illustrated below, find the number c such that the tangent lines to the graph at the points where $x = 1$ and $x = c$ are perpendicular.

• Find ~~tangent line~~ slope of tangent line at $x=1$
" " " " at $x=c$

At $x=1$: $f(x) = \frac{x^2}{3}$ $f'(x) = \frac{2x}{3} \Rightarrow f'(1) = \frac{2}{3} = m$

At $x=c$: $f'(x) = \frac{2x}{3} \Rightarrow f'(c) = \frac{2c}{3} = -\frac{1}{m}$

$$\frac{2c}{3} = \frac{-1}{\frac{2}{3}} \Rightarrow \frac{2c}{3} = \frac{-3}{2} \quad 4c = -9 \Rightarrow c = -\frac{9}{4}$$

~ slopes are perpendicular if this is satisfied.

This slide left intentionally blank

$$f(x) = \frac{(\sin(x))^2}{e^x + \sqrt{2}}$$

$$\bullet \sin^2 x + \cos^2 x = 1$$

Quotient rule: $(\text{numerator})' = (\sin x \cdot \sin x)'$
 $= \cos x \cdot \sin x + \sin x \cdot \cos x = 2 \sin x \cos x$

$$\begin{aligned} (f(x))' &= \left(\frac{\sin^2 x}{e^x + \sqrt{2}} \right)' = \frac{2 \sin x \cos x \cdot (e^x + \sqrt{2}) - \sin^2 x \cdot (e^x)}{(e^x + \sqrt{2})^2} \\ &= \frac{\sin x (2 \cos x (e^x + \sqrt{2}) - \sin x e^x)}{(e^x + \sqrt{2})^2} \end{aligned}$$

This slide left intentionally blank

This slide left intentionally blank