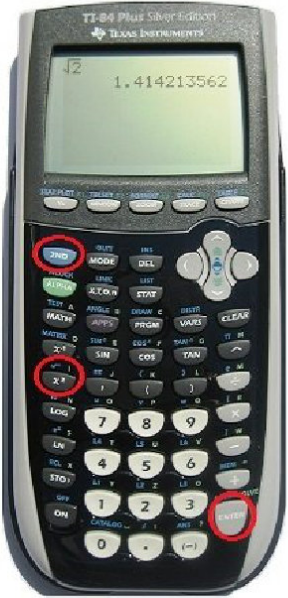
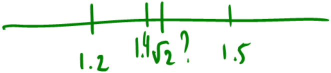
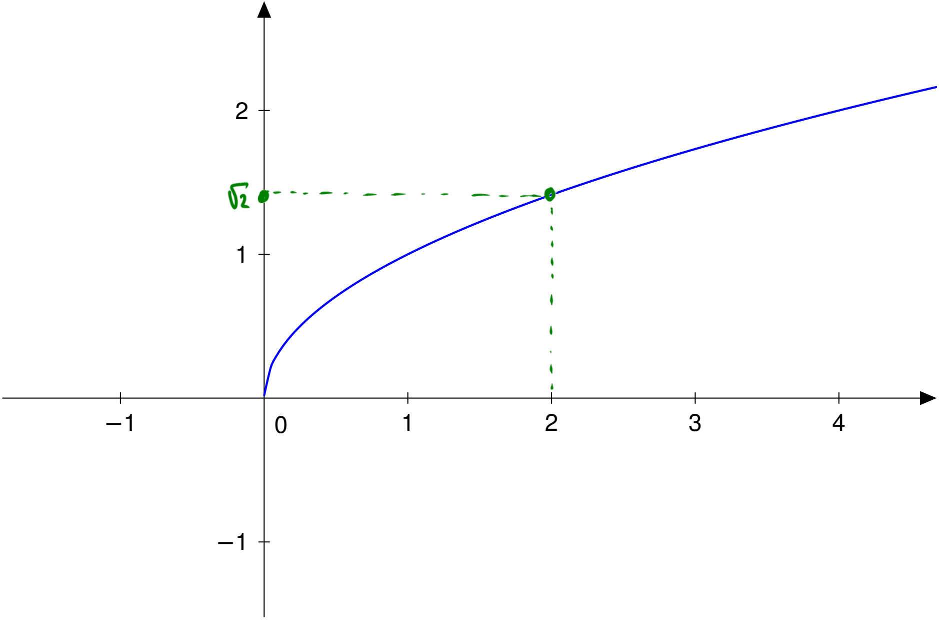
How does a calculator compute

$$\sqrt{2} \quad ?$$

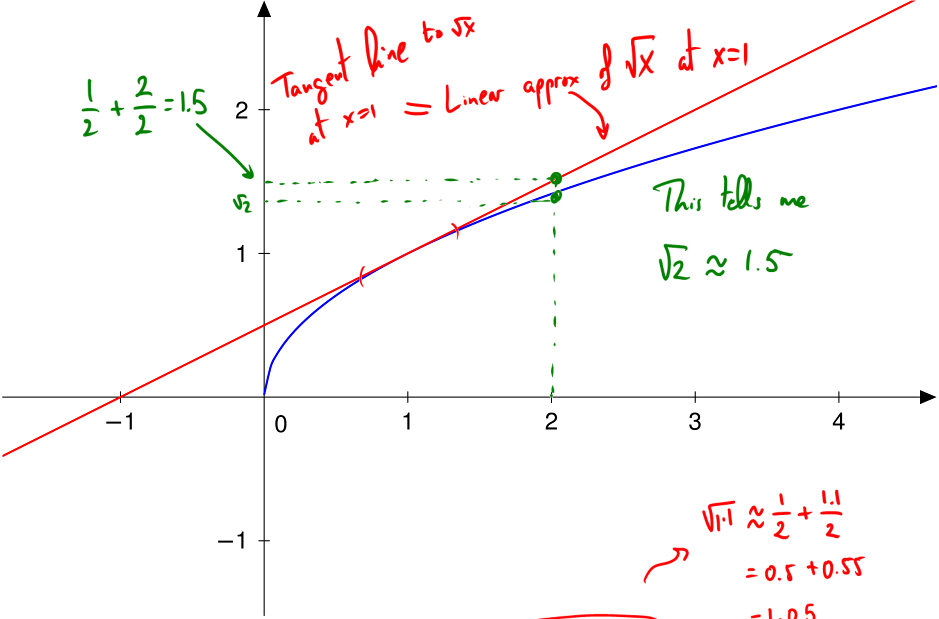


$\sqrt{2} = 1.2\ ?\quad \cancel{\Rightarrow}\quad 2 > (1.2)^2$





$y = \sqrt{x}$



Tangent line to $\sqrt{x}$ at $x=1$ = Linear approx of $\sqrt{x}$ at $x=1$

$\frac{1}{2}+\frac{2}{2}=1.5$

This tells me

$\sqrt{2} \approx 1.5$

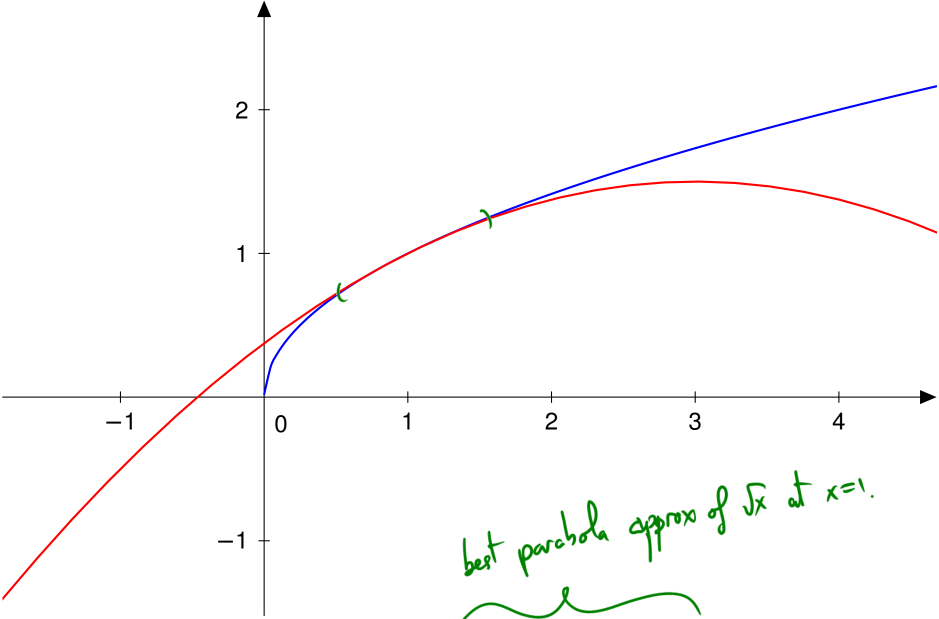
$\sqrt{1.1} \approx \frac{1}{2}+\frac{1.1}{2}$

$= 0.5 + 0.55$

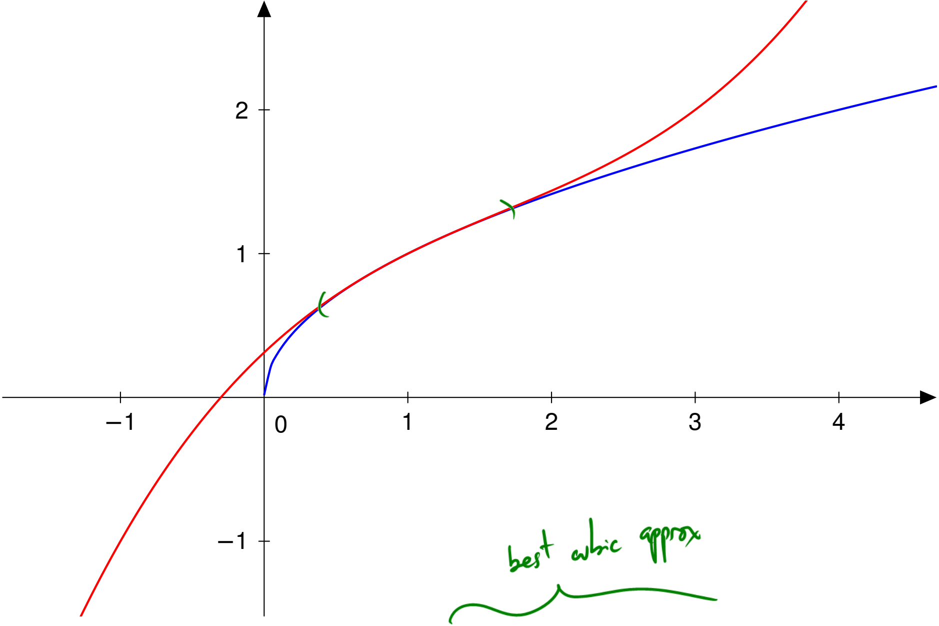
$= 1.05$

$\boxed{\sqrt{1.1} \approx 1.05}$

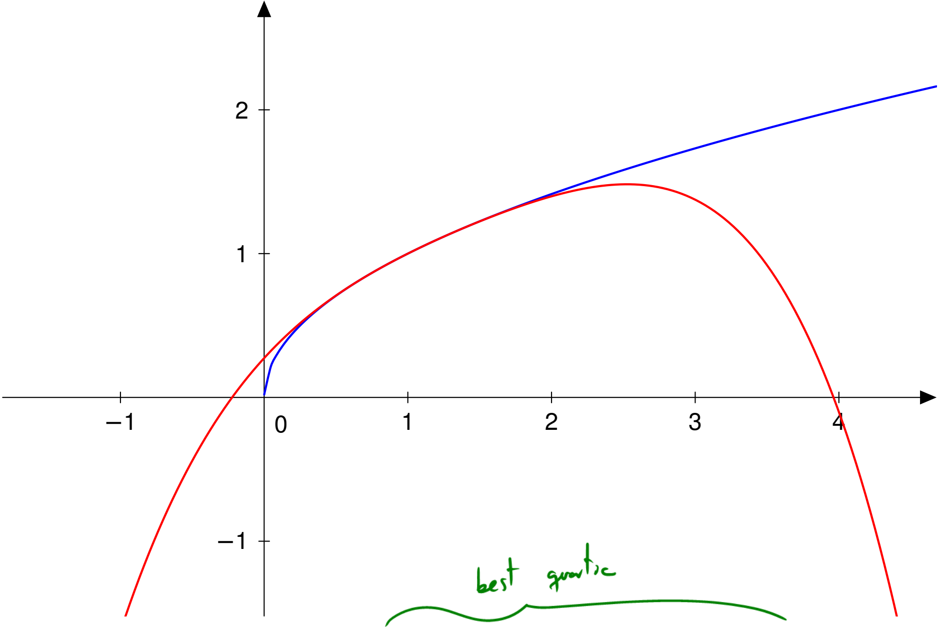
$y=\sqrt{x}$ and $\boxed{y=\frac{1}{2}+\frac{x}{2}}$



$$y = \sqrt{x} \quad \text{and} \quad y = \overbrace{\frac{3}{8} + \frac{3x}{4} - \frac{x^2}{8}}^{\text{best parabola approx of } \sqrt{x} \text{ at } x=1.}$$



$$y = \sqrt{x} \quad \text{and} \quad y = \overbrace{\frac{5}{16} + \frac{15x}{16} - \frac{5x^2}{16} + \frac{x^3}{16}}^{\text{best cubic approx}}$$



$$y = \sqrt{x} \quad \text{and} \quad y = \overbrace{\frac{35}{128} + \frac{35x}{32} - \frac{35x^2}{64} + \frac{7x^3}{32} - \frac{5x^4}{128}}^{\text{best quartic}}$$

Function	Value at 2
$\sqrt{x}$	$\sqrt{2} = 1.414213562373095\ldots$
$\frac{1}{2} + \frac{x}{2}$	$\frac{3}{2} = 1.5$ (error ≈ -0.085)
$\frac{3}{8} + \frac{3x}{4} - \frac{x^2}{8}$	$\frac{11}{8} = 1.375$ (error ≈ 0.039)
$\frac{5}{16} + \frac{15x}{16} - \frac{5x^2}{16} + \frac{x^3}{16}$	$\frac{23}{16} = 1.4375$ (error ≈ -0.023)
$\frac{35}{128} + \frac{35x}{32} - \frac{35x^2}{64} + \frac{7x^3}{32} - \frac{5x^4}{128}$	$\frac{179}{128} = 1.3984375$ (error ≈ -0.015)

MATH 1131Q - Calculus 1.

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Department of Mathematics
University of Connecticut

Day 17

Three Announcements

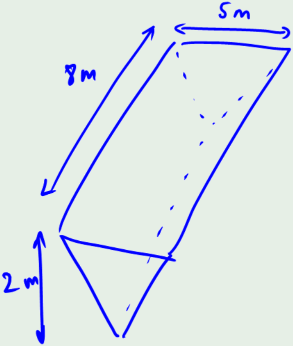
- **Office hours on Thursday are cancelled**. I'll have office hours Friday 1:30-2:30 instead.
- **Second midterm on Tuesday, November 4th**. More details later on.
- **Final Exam on Saturday, December 13th, 1-3pm.** More details later on.

Related Rates

Example

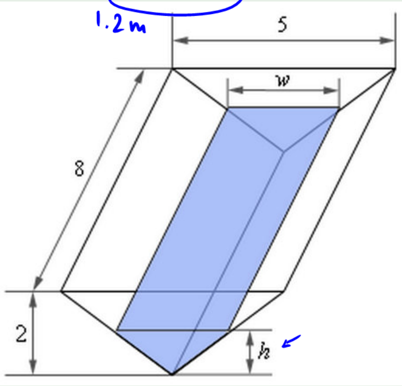
A trough of water is 8 meters deep and its ends are in the shape of isosceles triangles. The width of the trough is 5 meters and height is 2 meters. If water is being pumped in at a constant rate of 6 m^3/s. At what rate is the height of the water changing when the water has a height of 120 cm? (Source)

1.2 m.



Example

A trough of water is 8 meters deep and its ends are in the shape of isosceles triangles. The width of the trough is 5 meters and height is 2 meters. If water is being pumped in at a constant rate of 6 m^3/s. At what rate is the height of the water changing when the water has a height of 120 cm? (Source)



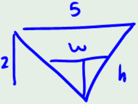
Want: $\frac{dh}{dt}$?

Variables: h, w, V

Data: $\frac{dV}{dt} = 6\ m^3/s$

Relations: $V = \frac{h \cdot w}{2} \cdot 8 = \frac{5h^2}{4} \cdot 8 = 10h^2$

Sim. Triangles:



$\frac{5}{2} = \frac{w}{h}$

$w = \frac{5h}{2}$

$V = 10h^2$

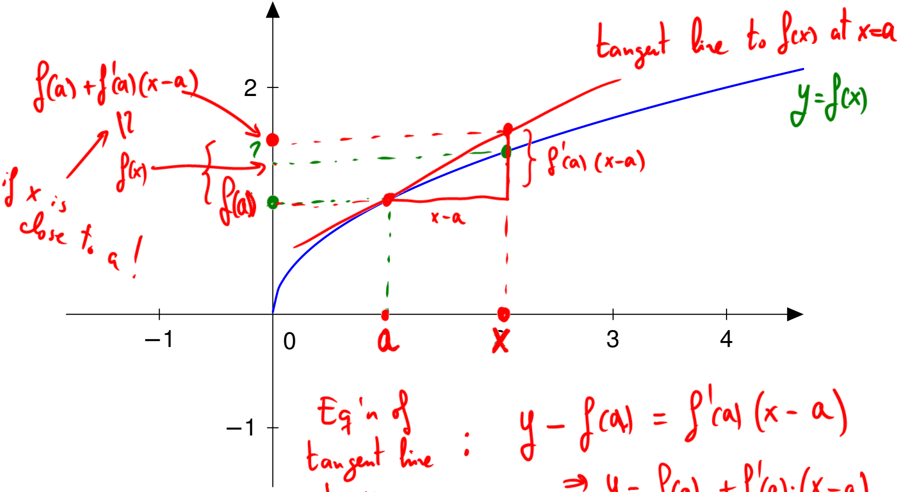
$$\frac{dV}{dt} = 6 = \frac{d}{dt}(10h^2) = 10 \cdot 2 \cdot h \cdot \frac{dh}{dt} = 20hh'$$

$$\Rightarrow h' = \frac{6}{20h} \Rightarrow_{\text{when } h=1.2} \boxed{h' = \frac{6}{20(1.2)} \text{ m/s}}$$

MORE

(Applications of Derivatives)

Linear Approximations

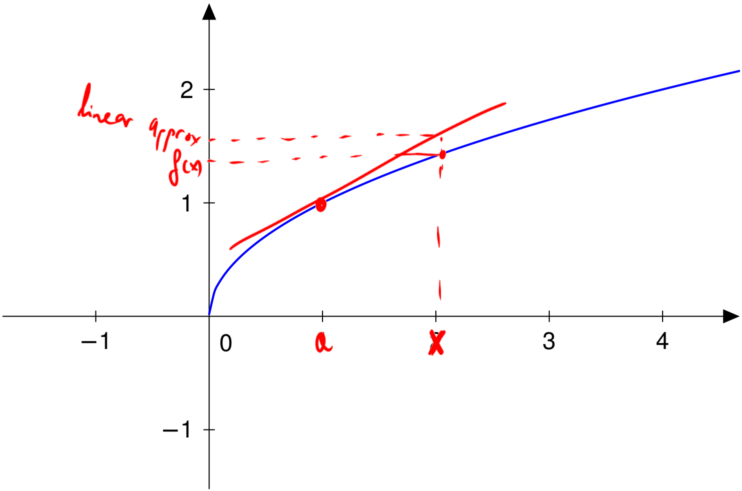


Eq'n of tangent line at x=a : $y - f(a) = f'(a)(x-a)$

(a, f(a)) slope f'(a)

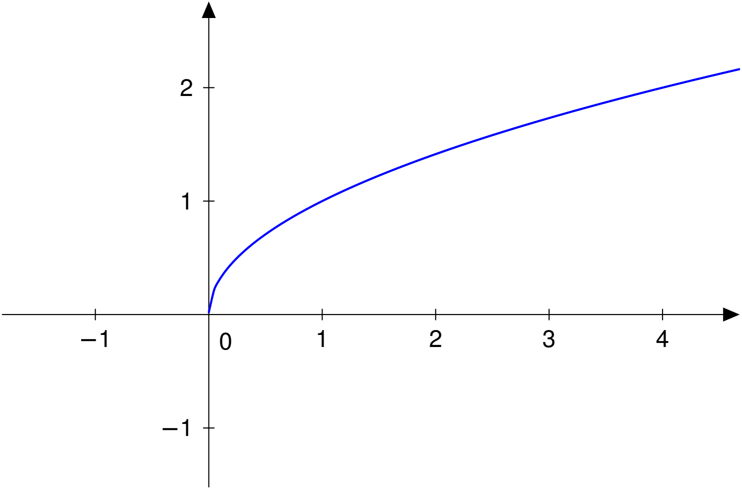
$\Rightarrow y = f(a) + f'(a)\cdot(x-a)$

Linear Approximations

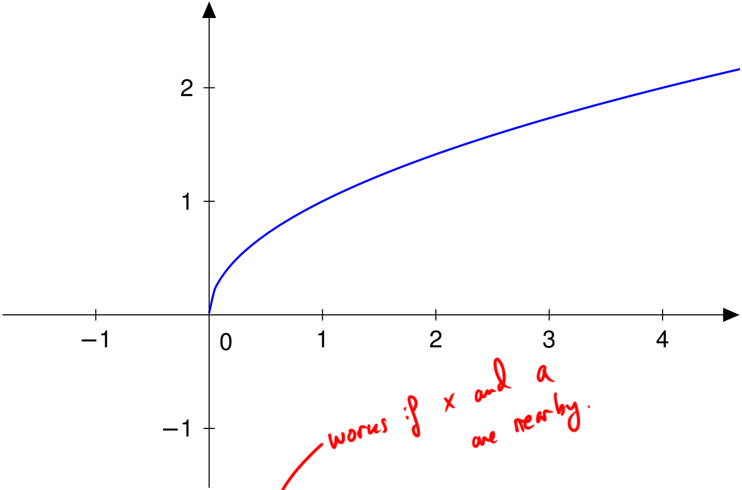


The approximation of $f(x)$ by its tangent line is called the **linear approximation** or **tangent line approximation** of f.

Linear Approximations



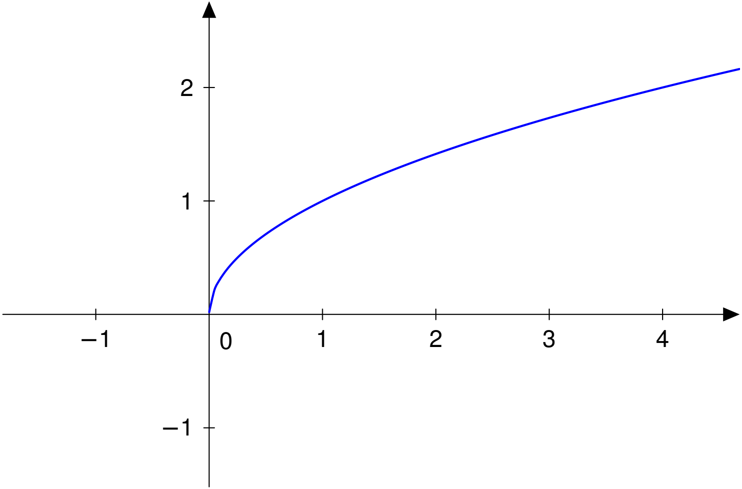
Linear Approximations



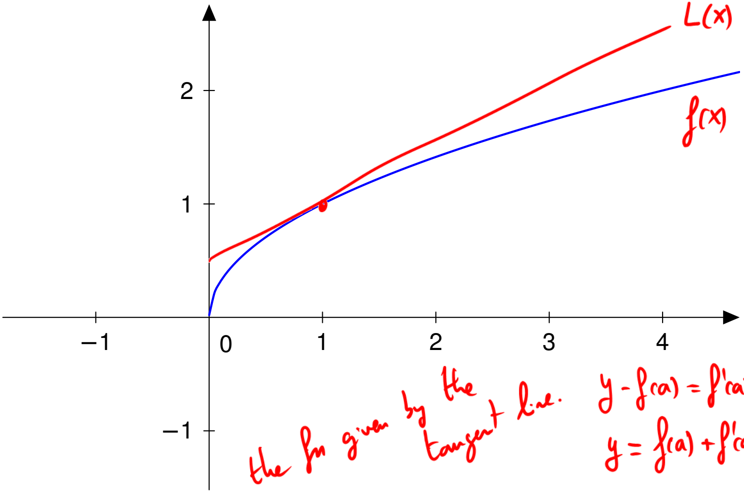
works if x and a are nearby.

The approximation of $\boxed{f(x) \approx f(a) + f'(a)(x-a)}$ is called the **linear approximation** of f at a.

Linear Approximations



Linear Approximations



$y - f(a) = f'(a)(x-a)$

$y = f(a) + f'(a)(x-a)$

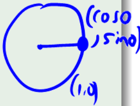
$\overbrace{\text{The function } L(x) = f(a) + f'(a)(x}^{\text{the fn given by the tangent line.}} - a)$ is called the **linearization** of f at a.

Example

Find the linearization of $y = \sin(x)$ at $a = 0$, and use it to find an approximate value of $\sin(0.5)$.

Linearization of $f(x) = \sin x$ at $a=0$ is given by

$$L(x) = f(a) + f'(a)(x-a)$$



- $a=0$
- $f(x) = \sin x \Rightarrow f(0) = 0$, $f'(x) = \cos x \Rightarrow f'(0) = \cos(0) = 1$

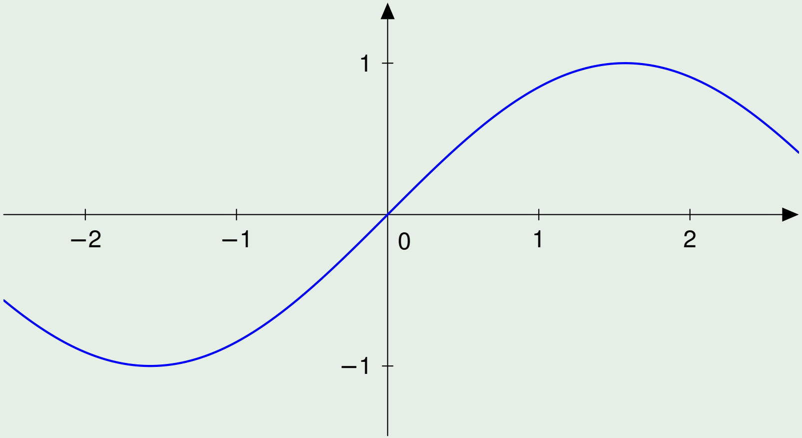
$$\Rightarrow L(x) = f(0) + f'(0)(x-0) = 0 + 1\cdot(x-0)$$

$$\Rightarrow \boxed{L(x) = x.}$$

Hence: $\sin(0.5) \approx L(0.5) = 0.5$ so $\boxed{\sin(0.5) \approx 0.5}$

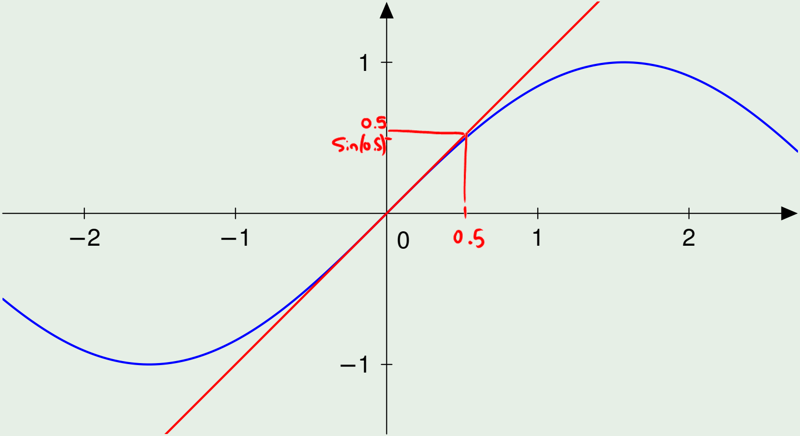
Example

Find the linearization of $y = \sin(x)$ at $a = 0$, and use it to find an approximate value of $\sin(0.5)$.



Example

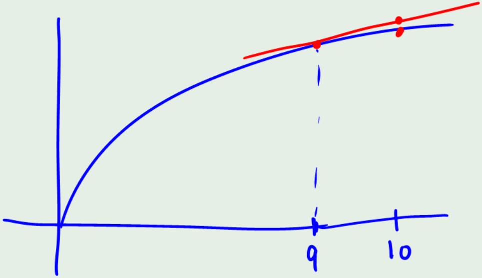
Find the linearization of $y = \sin(x)$ at $a = 0$, and use it to find an approximate value of $\sin(0.5) = 0.4794255386\ldots$.



Example

Use a linearization to approximate the value of $\sqrt{10}$.

$f(x) = \sqrt{x}$



$L(x) = f(a) + f'(a)(x-a)$

We are going to find the linear approx of $\sqrt{x}$ at $x=9$.

- $a = 9$
- $f(x) = \sqrt{x}$ $\quad f(9) = \sqrt{9} = 3$
- $f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

so $L(x) = 3 + \frac{1}{6}(x-9)$

so $\sqrt{10} \approx L(10) = 3 + \frac{1}{6}(10-9) = 3 + \frac{1}{6} = \frac{19}{6}$

Example

Use a linearization to approximate the value of $\sqrt{10}$.

$$\sqrt{10} = 3.162277660168\ldots \qquad \longleftarrow$$

$$L(10) = \frac{19}{6} = 3.166666666666\ldots \qquad \longleftarrow$$

$$\text{error} = \sqrt{10} - L(10) = -0.004389006498\ldots$$

(Alternately: linearized $\sqrt{9+x}$ at $a=0$.

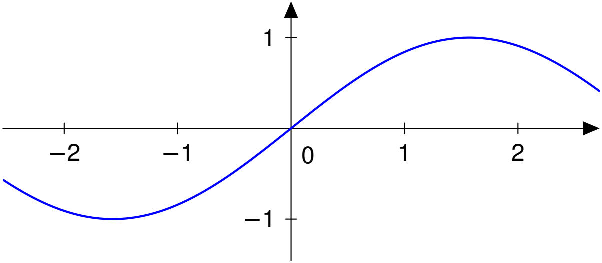
$\sqrt{10} \approx L(1)$)

Maximum and Minimum Values of a Function

Definition

Let c be a number in the domain D of a function f. Then $f(c)$ is the

1. **absolute maximum** *value of f on D if $f(c) \geq f(x)$ for all x in D.*
2. **absolute minimum** *value of f on D if $f(c) \leq f(x)$ for all x in D.*



The function $f(x) = \sin x$ in $\mathbb{R}$ has an absolute maximum value of 1 and an absolute minimum value of -1. These are attained as $f(\pi/2) = 1$ and $f(-\pi/2) = -1$.

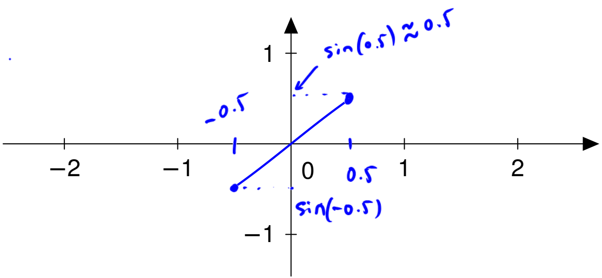
Maximum and Minimum Values of a Function

Definition

Let c be a number in the domain D of a function f. Then $f(c)$ is the

1. **absolute maximum** *value of f on D if $f(c) \geq f(x)$ for all x in D.*
2. **absolute minimum** *value of f on D if $f(c) \leq f(x)$ for all x in D.*

also global.



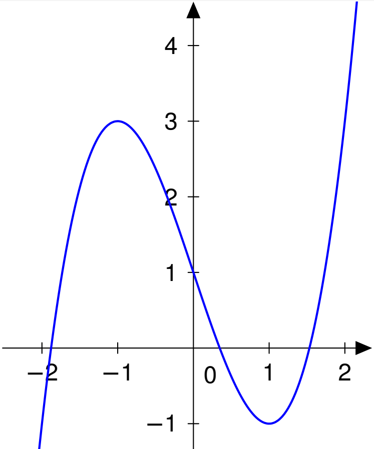
The function $f(x) = \sin x$ in $[-0.5, 0.5]$ has an absolute maximum value of $\sin(0.5)$ and an absolute minimum value of $\sin(-0.5)$. These are attained as $f(0.5) = \sin(0.5)$ and $f(-0.5) = \sin(-0.5)$.

Maximum and Minimum Values of a Function

Definition

Let c be a number in the domain D of a function f. Then $f(c)$ is the

1. **local maximum** *value of f if $f(c) \geq f(x)$ when x is near c.*
2. **local minimum** *value of f if $f(c) \leq f(x)$ when x is near c.*



The function

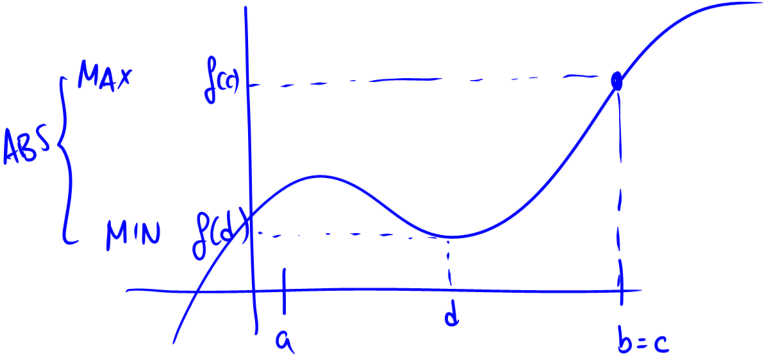
$$f(x) = x^3 - 3x + 1$$

has a local maximum value of 3 at $c = -1$, and a local minimum value of -1 at $c = 1$.

Maximum and Minimum Values of a Function

Theorem (Extreme Value Theorem)

If f is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in [a, b].

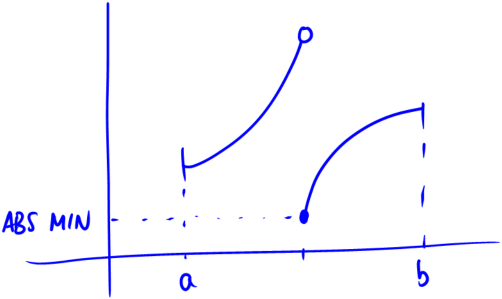


Maximum and Minimum Values of a Function

Theorem (Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

WARNING! Continuity is essential in the extreme value theorem.

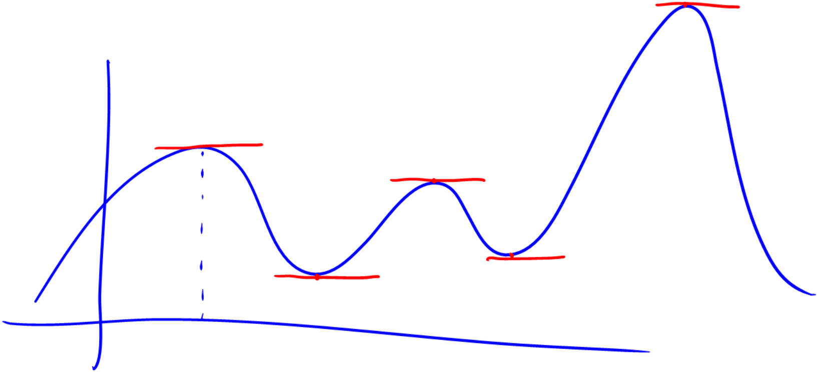


No <u>abs</u> MAX in this fn.

Maximum and Minimum Values of a Function

Theorem (Fermat's Theorem)

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.



Maximum and Minimum Values of a Function

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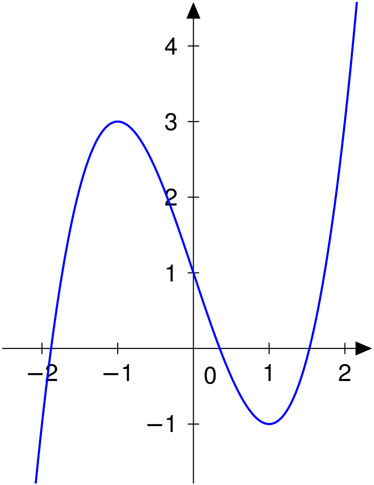


The function

$$f(x) = x^3 - 3x + 1$$

has a local maximum value of 3 at $c = -1$, and a local minimum value of -1 at $c = 1$.

Check $f'(x) = 3x^2 - 3$

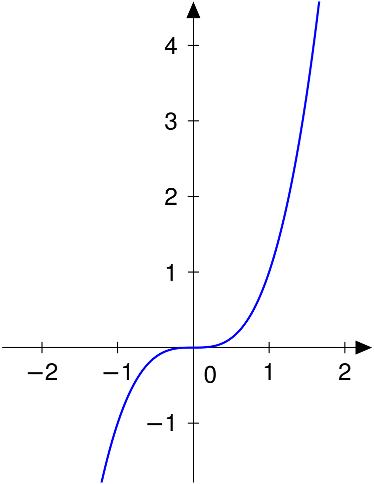
$f'(-1) = 3 \cdot (-1)^2 - 3 = 0$ ✓

$f'(1) = 3 \cdot 1^2 - 3 = 0$ ✓

Maximum and Minimum Values of a Function

Theorem (Fermat's Theorem)

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.



WARNING! The converse is not necessarily true!

The function

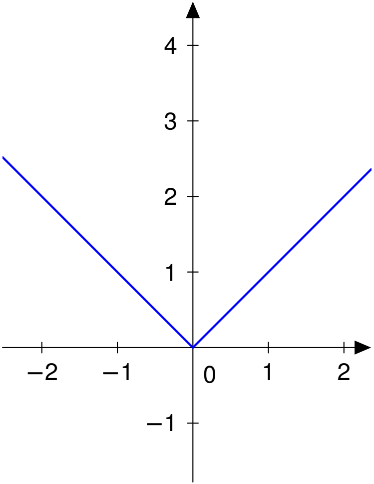
$$f(x) = x^3 \qquad f'(x) = 3x^2$$

satisfies $f'(0) = 0$, but there is no local max or min at 0.

Maximum and Minimum Values of a Function

Theorem (Fermat's Theorem)

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.



WARNING! A local or absolute max/min may occur at places where f' is **not defined!**

The derivative of the function

$$f(x) = |x|$$

is undefined at 0, but there is an absolute minimum at 0.

Maximum and Minimum Values of a Function

Theorem (Fermat's Theorem)

If $f(x)$ has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Definition

A **critical number** *of a function $f(x)$ is a number c in the domain of f such that (1) $f'(c) = 0$, or (2) $f'(c)$ does not exist.*

Example

Find the critical points of $f(x) = x^3 - 3x + 1$.

$f'(x) = 3x^2 - 3 \longrightarrow$ defined everywhere!

$f'(x) = 0$? $3x^2 - 3 = 0 \Rightarrow 3x^2 = 3$

$\Rightarrow x^2 = 1$

$\Rightarrow \boxed{x = \pm 1}$

Critical pts: $x = 1$ & $x = -1$.

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