And now for some bonus points, since I need them from my last exam, here is a picture of Batman.

"And now for some bonus points, since I need them from my last exam, here is a picture of Batman."

Batman: "I deserve 10 extra points".
Our second Midterm Exam is TODAY!

- Two times (check your time following these instructions)
  1. 6-8 PM at AUST 108, or
  2. 9-11 PM at TLS 154.

- There is a practice exam and solutions in outline in the website:
  http://alozano.clas.uconn.edu/math1131f14

- Covers Sections 1.1 - 4.7 with special emphasis on new material: 3.4 - 4.7.

- Today’s class: review.

- There was a review session on Monday 9/29, now available here (click on Calculus I (MATH 1131) Exam II Review).
Proof that I am not David Daggett.
Midterm 2:

Derivatives (Chain Rule and Implicit Differentiation)

Applications: Exponential Growth/Decay

Applications: Related Rates

Applications: Extreme Values

Applications: Sketching Graphs

Applications: Optimization
Clicker Question: I struggle the most with...

(A) Using the chain rule to find a derivative and/or implicit differentiation.

(B) Finding an exponential model of growth/decay.

(C) Related rates problems.

(D) Sketching the graph of a function.

(E) Optimization problems.
Derivatives
(Rates of Change)
Let $f(x)$ and $g(x)$ be differentiable functions (i.e., $f'(x)$ and $g'(x)$ exist at every point $x$).

**Derivatives**

- $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.
- $(c)' = 0$, where $c$ is a constant.
- $(x^n)' = nx^{n-1}$, where $n$ is real.
- $(a^x)' = ka^x$, where $a > 0$, $k = \lim_{h \to 0} \frac{a^h - 1}{h}$.
- $(e^x)' = e^x$.
- $(\sin x)' = \cos x$.
- $(\cos x)' = -\sin x$.

**Rules**

- $(cf)' = cf'$, where $c$ is a constant.
- $(f + g)' = f' + g'$.
- Product rule: $(fg)' = f'g + fg'$.
- Quotient rule: \( \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \).
- Chain rule: \( (f(g(x)))' = f'(g(x))g'(x) \).
Example (Practice Test, Problem 1(b))

If \( f(x) \) is a differentiable function, then the derivative of \( f(e^x) \) is \( f'(e^x) e^x \). True or False, with justification.

\[
(f(e^x))' = f'(e^x) \cdot e^x
\]

\[
(f(g(x)))' = f'(g(x)) \cdot g'(x).
\]
Example (Practice Test, Problem 2(a))

Compute \( \frac{d}{dx} \left( x \ln^2(x) \right) \). You do not need to simplify.

1) Prod rule → 2) Chain rule

\[
\left( x \ln^2 x \right)' = 1 \cdot \ln^2 x + x \left( 2 \cdot \ln x \cdot \frac{1}{x} \right)
\]
Example (Practice Test, Problem 2(b))

Find the derivative of $y = 5^x$ in two ways and check they agree:

(i) logarithmic differentiation, writing the final answer entirely in terms of $x$.

(ii) express $5^x$ as a power of $e$ and use the fact that $(e^x)' = e^x$.

\[
\begin{align*}
(i) & \quad y = 5^x \Rightarrow \ln y = \ln 5^x = x \ln 5 \\
& \quad \Rightarrow \frac{y'}{y} = \ln 5 \quad \Rightarrow \quad y' = y \cdot \ln 5 = 5^x \cdot \ln 5 \\
(ii) & \quad 5^x = (e^{\ln 5})^x = e^{x \ln 5} \\
& \quad \Rightarrow (5^x)' = (e^{x \ln 5})' = e^{x \ln 5} \cdot \ln 5 = 5^x \cdot \ln 5
\end{align*}
\]
Example (Practice Test, Problem 3)

Use implicit differentiation to find the equation of the tangent line to the graph of \( y^2 = x^3 + 2xy \) at the point \((3, -3)\), as marked below. Write the equation in the form \( y = mx + b \).
Use implicit differentiation to find the equation of the tangent line to the graph of \( y^2 = x^3 + 2xy \) at the point \((3, -3)\), as marked below. Write the equation in the form \( y = mx + b \).

\[
\frac{d}{dx} \quad y^2 = x^3 + 2xy \quad \Rightarrow \quad 2yy' = 3x^2 + 2y + 2x \cdot y' \\
\Rightarrow \quad y' = \frac{3x^2 + 2y}{2y - 2x} \quad \Rightarrow \quad y'(3, -3) = \frac{3 \cdot 3^2 + 2 \cdot (-3)}{2 \cdot (-3) - 2 \cdot 3} = \frac{3 \cdot 9 - 6}{-6 - 6} = \frac{21}{-12} = -\frac{7}{4}
\]

Tangent: \( y + 3 = m(x - 3) \)

\[
\Rightarrow \quad y = mx - 3m - 3
\]
Linear Approximations
The approximation of \( f(x) \) by its tangent line is called the **linear approximation** or **tangent line approximation** of \( f \).

The approximation of \( f(x) \approx f(a) + f'(a)(x - a) \) is called the **linear approximation** of \( f \) at \( a \).

The function \( L(x) = f(a) + f'(a)(x - a) \) is called the **linearization** of \( f \) at \( a \).
Example (Practice Test, Problem 4)

(a) Find the linearization of \( \ln(x) \) at \( a = 1 \). (b) Use part (a) to find an approximate value of \( \ln(2) \). (You must use part (a). A calculator value of \( \ln(2) \) will earn 0 points.)

\[
\begin{align*}
\mathbf{f}(x) &= \ln(x), \quad \mathbf{f}(1) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{f}(x) = \frac{1}{x}, \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{f}(1) = \frac{1}{1} = 1 \\
L(x) &= 0 + 1 \cdot (x - 1) = x - 1
\end{align*}
\]

then \( \ln(x) \approx L(x) \) for \( x \) near \( a = 1 \)

\[\ln(2) \approx L(2) = 2 - 1 = 1 \quad \text{so} \quad \ln 2 \approx 1.\]

- The approximation of \( f(x) \approx f(a) + f'(a)(x - a) \) is called the linear approximation of \( f \) at \( a \).
- The function \( L(x) = f(a) + f'(a)(x - a) \) is called the linearization of \( f \) at \( a \).
Exponential Growth and Decay
**Definition**

We say that a function $f(x)$ grows (or decays) exponentially if the growth rate at any $x$ is proportional to the function’s value.

In other words, $f'(x)$ is proportional to $f(x)$, or equivalently, there is a constant $k$ such that

$$f'(x) = k \cdot f(x), \quad \text{or} \quad y' = k \cdot y, \quad \text{or} \quad \frac{dy}{dx} = k \cdot y.$$  

**Question**

What functions satisfy this **differential equation**?

**Theorem**

The only solutions of the differential equation $y' = ky$ is the family of exponential functions

$$y = Ce^{kx}.$$  

Note: $y(0) = Ce^{k \cdot 0} = Ce^0 = C \cdot 1 = C$. Thus, $C = y(0)$. 
Example (Practice Test, Problem 6)

A pile of the radioactive substance Unobtainium loses 6% of its mass in a year.

1. If a sample of Unobtainium has an initial mass of 50 grams, determine a formula for \( U(t) \), the amount of Unobtainium left in the sample after \( t \) years.

2. Find the half-life of Unobtainium in years, accurate to 3 decimal places.

\[
U(t) = C \cdot e^{kt}, \quad C = U(0) = 50 \text{g}.
\]

(1) \( U(1) = \frac{50 \cdot 0.94}{100} = 47 \text{g} = 50 \cdot e^{k \cdot 1} \Rightarrow 50 e^{k} = 47 \Rightarrow e^{k} = \frac{47}{50} \Rightarrow k = \ln(\frac{47}{50}) \Rightarrow U(t) = 50 \cdot e^{\ln(\frac{47}{50}) \cdot t} \)

(2) \( 25 = U(t) = 50 e^{\ln(\frac{47}{50}) \cdot t} \Rightarrow e^{\ln(\frac{47}{50}) \cdot t} = \frac{1}{2} \Rightarrow \ln(\frac{47}{50}) \cdot t = \ln(\frac{1}{2}) \Rightarrow t = \frac{\ln(\frac{1}{2})}{\ln(\frac{47}{50})} \).
Example (Practice Test, Problem 6)

A pile of the radioactive substance Unobtainium loses 6% of its mass in a year.

1. If a sample of Unobtainium has an initial mass of 50 grams, determine a formula for $U(t)$, the amount of Unobtainium left in the sample after $t$ years.

2. Find the half-life of Unobtainium in years, accurate to 3 decimal places.

Done!
Related Rates
I hate math tests because all through the chapter it's like really easy and then you think you've got it and then the test is like

IF I THROW A TRIANGLE OUT OF THE CAR AND THE CAR IS GOING 20KM/H AND WIND RESISTANCE IS A THING THAT EXISTS, HOW MANY CUPCAKES CAN PEDRO BUY WITH ONE HUMAN SOUL

As seen on Reddit.
Related Rates

The most common way to approach related rates problems is the following:

1. **Identify the known variables**, including rates of change and the rate of change that is to be found. *(Drawing a picture* or representation of the problem can help to keep everything in order)*

2. Construct an **equation relating the quantities** whose rates of change are known to the quantity whose rate of change is to be found.

3. **Differentiate both sides** of the equation with respect to time (or other rate of change). Often, the chain rule is employed at this step.

4. **Substitute the known rates** of change and the known quantities into the equation.

5. **Solve for the wanted rate of change**.
Related Rates

We know this rate

rate A

If we knew the relationship between rate A and rate B, we could deduce rate B. How can we find this relationship?

differentiate

position A

The relationship between position A and position B is known. Differentiate. This will provide the relationship we need.

rate B

We want to know this rate

differentiate

position B
Example (Practice Test, Problem 5(a))

Two boats leave a dock at the same time. One boat travels south at 30 mi/hr and the other travels west at 40 mi/hr. After half an hour, how quickly is the distance between the boats increasing, in mi/hr?
A spy plane is flying 500 m above the ground at 450 km/h, and its path goes directly over an enemy tracking station that is already tracking it. 

(i) How many meters does the plane cover in two seconds? 

(ii) Determine how quickly the angle between the ground and the line from the tracking station to the plane is changing, in radians per second, two seconds after the plane flies over the tracking station.
Example (Practice Test, Problem 5(b))

A spy plane is flying 500 m above the ground at 450 km/h, and its path goes directly over an enemy tracking station that is already tracking it.

(i) How many meters does the plane cover in two seconds?

(ii) Determine how quickly the angle between the ground and the line from the tracking station to the plane is changing, in radians per second, two seconds after the plane flies over the tracking station.

IT’S A TRAP!
Example (Practice Test, Problem 5(b))

A spy plane is flying 500 m above the ground at 450 km/h, and its path goes directly over an enemy tracking station that is already tracking it.

(i) How many meters does the plane cover in two seconds?

(ii) Determine how quickly the angle between the ground and the line from the tracking station to the plane is changing, in radians per second, two seconds after the plane flies over the tracking station.

\[ (450 \text{ km/h} = 450 \cdot 1000/3600 \text{ m/s} = 125 \text{ m/s}) \]
Applications of Derivatives: Extreme Values
Definition

Let $c$ be a number in the domain $D$ of a function $f$. Then $f(c)$ is the

1. **absolute maximum** value of $f$ on $D$ if $f(c) \geq f(x)$ for all $x$ in $D$.
2. **absolute minimum** value of $f$ on $D$ if $f(c) \leq f(x)$ for all $x$ in $D$.
3. **local maximum** value of $f$ if $f(c) \geq f(x)$ when $x$ is near $c$.
4. **local minimum** value of $f$ if $f(c) \leq f(x)$ when $x$ is near $c$.

Theorem (Extreme Value Theorem)

If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.

Theorem (Fermat’s Theorem)

If $f$ has a local maximum or minimum at $c$, and if $f'(c)$ exists, then $f'(c) = 0$. 

Definition

A critical number of a function $f(x)$ is a number $c$ in the domain of $f$ such that (1) $f'(c) = 0$, or (2) $f'(c)$ does not exist.

How to find the absolute maximum and minimum values of a function $f(x)$ on a closed interval $[a, b]$:

1. Find the critical values of $f$ in $[a, b]$. Calculate the values of $f$ at all critical values.

2. Find the values of $f$ at endpoints of the interval and at any point where $f$ is defined but not continuous.

3. The largest of the values from (1) and (2) is the absolute maximum value; the smallest of these values is the absolute minimum value.
Use calculus to find the absolute maximum and minimum values of the following functions on the indicated intervals. Answers can be given to three decimal places.

(a) $f(x) = \sin x + \cos x$, on $[0, \pi]$

1) Critical points: 
\[
\frac{d}{dx}(x) = 0 = \cos x - \sin x \\
\text{DNE? Never.}
\]

2) End points: 
\[
\begin{align*}
&x = 0 \\
&x = \pi
\end{align*}
\]

3) Compare:
\[
\begin{align*}
f(0) &= 1 \\
f(\pi) &= -1 \\
&f\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}} + \frac{\sqrt{2}}{2} = 2 \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\
\end{align*}
\]

\[
\begin{align*}
\text{ABS MAX} &= \sqrt{2} \text{ at } x = \frac{\pi}{4} \\
\text{ABS MIN} &= -1 \text{ at } x = \pi.
\end{align*}
\]
Use calculus to find the absolute maximum and minimum values of the following functions on the indicated intervals. Answers can be given to three decimal places.

(b) $f(x) = (7x - 1)e^{-2x}$, on $[0, 1]$
Sketching Graphs of Functions and l’Hospital’s Rule
Using the First Derivative

Theorem (Increasing/Decreasing Test)

Let \( f \) be a differentiable function on an interval \( (a, b) \).

1. If \( f'(x) > 0 \) on \( (a, b) \), then \( f \) is increasing on \( (a, b) \).
2. If \( f'(x) < 0 \) on \( (a, b) \), then \( f \) is decreasing on \( (a, b) \).

Theorem (The First Derivative Test for Critical Points)

Suppose that \( c \) is a critical number of a continuous function \( f \). Then:

1. If \( f' \) changes from positive to negative at \( c \), then \( f \) has a local maximum at \( c \).
2. If \( f' \) changes from negative to positive at \( c \), then \( f \) has a local minimum at \( c \).
3. If \( f' \) does not change sign at \( c \), then \( f \) has no local maximum or minimum at \( c \).
### Using the Second Derivative

#### Theorem (Concavity Test)

*Let $f$ be a twice differentiable function on an interval $(a, b)$.*

1. If $f''(x) > 0$ on $(a, b)$, then $f$ is concave upward on $(a, b)$.
2. If $f''(x) < 0$ on $(a, b)$, then $f$ is concave downward on $(a, b)$.

#### Definition

*A point $P$ on a curve $y = f(x)$ is called an inflection point if $f$ is continuous there and the curve changes concavity at $P$.*

#### Theorem (The Second Derivative Test for Critical Points)

*Suppose that $c$ is a critical number of a continuous function $f$, and suppose $f''$ is continuous near $c$. Then:*

1. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive.
Example (Practice Test, Problem 8)

When \( f(x) = \frac{x}{x^2 + 1} \), use calculus to find

(i) the critical numbers of \( f(x) \),
Example (Practice Test, Problem 8)

When \( f(x) = \frac{x}{x^2 + 1} \), use calculus to find

(ii) the open intervals where \( f(x) \) is increasing and where \( f(x) \) is decreasing,
Example (Practice Test, Problem 8)

When \( f(x) = \frac{x}{x^2 + 1} \), use calculus to find

(iii) the open intervals where the graph of \( y = f(x) \) is concave up and concave down.
Sketch the graph of $f(x) = \frac{x}{x^2 + 1}$. 
Sketch the graph of \( f(x) = \frac{x}{x^2 + 1} \).
l’Hospital Rule
The following types of limits are said to be of indeterminate form:

\[
\begin{align*}
0 & \quad \infty \\
0 & \quad \infty' \\
\infty & \quad \infty \\
\infty & \quad \infty' \\
\infty & \quad 0 \\
\infty & \quad 0 \\
1 & \quad \infty
\end{align*}
\]

For example,

\[
\begin{align*}
\lim_{x \to 1} \frac{\sin x}{x}, \\
\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1}, \\
\lim_{x \to \infty} x^2 e^{-x}, \\
\lim_{x \to \infty} \sqrt{x^2 + 1 - x}, \\
\lim_{x \to 0^+} x^x, \\
\lim_{x \to \infty} x^{1/x}, \\
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.
\end{align*}
\]

These limits need to be manipulated, using algebraic reasoning, or other methods such as... the l’Hospital Rule.
Guillaume François Antoine, Marquis de l’Hospital (1661-1704) was a French mathematician.
l’Hospital Rule (for $0/0$ or $\infty/\infty$ indeterminate forms)

**Theorem**

Let $f$ and $g$ be differentiable functions on an open interval $I$ except possibly at a point $c$ in $I$. Moreover, assume that

1. \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \) or $\pm \infty$, and

2. \( \lim_{x \to c} \frac{f'(x)}{g'(x)} \) exists, and $g'(x) \neq 0$ for all $x$ in $I$ with $x \neq c$.

Then:

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.
\]

Note: here $I$ can be an infinite interval and $c$ could be $\pm \infty$. 
Evaluate the following limits using l’Hospital’s Rule.

(a) \[ \lim_{{x \to 0}} \frac{5^x - 4^x}{3^x - 2^x} \]
Evaluate the following limits using l’Hospital’s Rule.

(b) \( \lim_{x \to 0} \frac{\sin^2(ax)}{x^2} \), where \( a \neq 0 \). (The answer will depend on \( a \).)
Example (Practice Test, Problem 11)

Evaluate the following limits using l’Hospital’s Rule.

(c) \( \lim_{x \to \infty} \left( 1 + \frac{10}{x^2} \right)^{x^2} \)
Optimization
How to approach optimization problems:

1. **Understand the problem.** What is the unknown? What are the given variables? What data is given?
2. **Draw a diagram** relating the variables of the problem.
3. **Assign names** to the variables, and find equations among the variables. Let’s say $Q$ is the variable to be optimized.
4. **Express the optimization variable $Q$ in terms of the other variables.**
5. If the optimization variable $Q$ is expressed in terms of several variables, use the equations relating the variables to **express $Q$ in terms of one single variable $x$.**
6. **Find the absolute maximum or minimum of the optimization variable $Q(x)$**, using our extreme values methods, **in a sensible interval.**
Example

Find the point on the parabola \( y^2 = 2x \) that is closest to the point \((1, 4)\).

\[
d = \sqrt{(x-1)^2 + (y-4)^2}
\]

Minimize \( d \), \( y \)

\[
Min \ g \quad d^2 = D = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 \quad \text{in} \ (-\infty, \infty)
\]

\[
= \frac{y^4}{4} + 1 - y^2 + \frac{y^2}{2} - 8y + 16 = \frac{y^4}{4} - 8y + 17
\]

\[
D' = y^3 - 8 \Rightarrow \text{critical points: } y^3 - 8 = 0 \Rightarrow y = 2 \Rightarrow y = 2
\]

\[
D'' = 3y^2 > 0 \quad \text{always positive, 2nd derivative test } \Rightarrow \text{local min} \Rightarrow \text{absolute min}
\]

\[
y = 2
\]
Example

A box-shaped shipping crate with a square base needs to have a volume of 80ft$^3$. The material used to make the base of the crate costs twice as much (per ft$^2$) as the material used for the sides, and the material used to make the top of the crate costs half as much (per ft$^2$) as the material used for the sides. Use calculus to find the dimensions of the crate that minimize the total cost of the materials.