Calculus Trivia: Historic Calculus Texts

- **Archimedes of Syracuse** (**c. 287 BC - c. 212 BC**) - "*On the Measurement of a Circle*": Archimedes shows that the value of pi (π) is greater than 223/71 and less than 22/7 using rudimentary calculus.

- **Jyeshtadeva** - "*Yuktibhasa*": Written in India in **1501**, this was the world's first calculus text.

- **Gottfried Leibniz** - "*Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illi calculi genus*": Published in **1684**, this is Leibniz's first treaty on differential calculus.

- **Isaac Newton** - "*Philosophiae Naturalis Principia Mathematica*": This is a three-volume work by Isaac Newton published on 5 July **1687**. Perhaps the most influential scientific book ever published.

MATH 1131Q - Calculus 1.

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Day 6

Limits

Previously...

Definition (Informal Definition of Limit)

Let $f(x)$ be a function that is defined when x is near the number a (i.e., f is defined on some open interval that contains a, except possibly a itself). Then, we write

$$\lim_{x \to a} f(x) = L$$

and we say the limit of $f(x)$, as x approaches a, equals L, if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Also:

- Sided limits: $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$,
- Infinite limits: $\lim_{x \to a} f(x) = \infty$, and
- The Limit Laws.

The FORMAL Definition of Limit

Definition (Formal Definition of Limit)

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then, we say that the limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } \; 0 < |x - a| < \delta, \quad \text{then } \; |f(x) - L| < \varepsilon.$$

Definition (Formal Definition of Limit)

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ *there is a number* $\delta > 0$ *such that*

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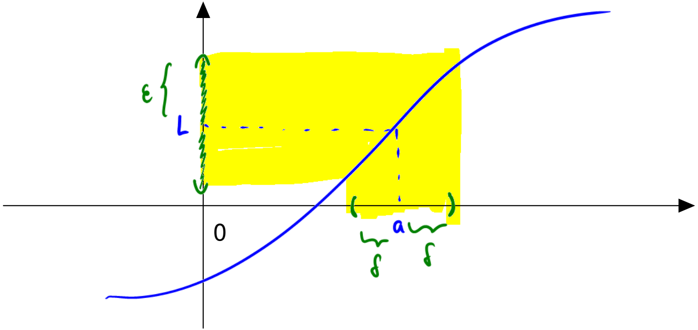

The FORMAL Definition of Sided Limits

Definition (Formal Definition of Sided Limit)

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then, we say that

1. *the limit of $f(x)$ as x approaches a* from the left *is L, and we write $\lim_{x\to a^-} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that*
$$\textit{if } a-\delta < x < a, \quad \textit{then } |f(x)-L| < \varepsilon.$$
2. *the limit of $f(x)$ as x approaches a* from the right *is L, and we write $\lim_{x\to a^+} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that*
$$\textit{if } a < x < a+\delta, \quad \textit{then } |f(x)-L| < \varepsilon.$$

The FORMAL Definition of Infinite Limits

Definition (Formal Definition of Infinite Limit)

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then, we say that

1. *the limit of $f(x)$ as x approaches a* from the left *is infinite, and we write* $\lim_{x\to a} f(x) = \infty$ *if for every number $M > 0$ there is a number $\delta > 0$ such that*

$$\text{if } \; 0 < |x - a| < \delta, \quad \text{then } \; f(x) > M.$$

The FORMAL Definition of Infinite Limits

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2. *the limit of $f(x)$ as x approaches a* from the left *is minus infinity, and we write* $\lim_{x\to a} f(x) = -\infty$ *if for every number $M > 0$ there is a number $\delta > 0$ such that*

$$\textit{if } \; 0 < |x - a| < \delta, \quad \textit{then } \; f(x) < -M.$$

The FORMAL Definition of Infinite Limits

Definition (Formal Definition of Infinite Limit)

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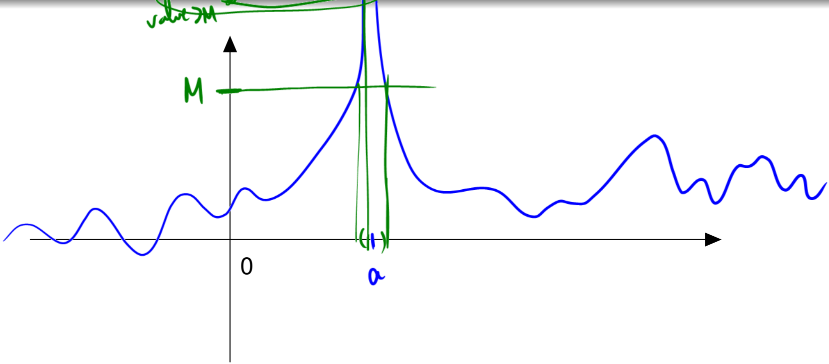
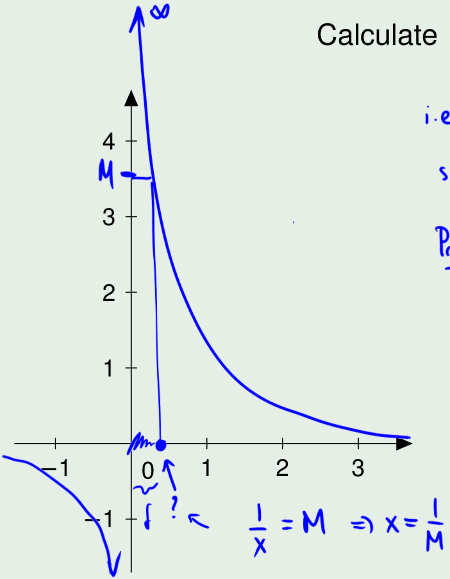


Example



Calculate $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

i.e., for all $M>0$ there is a $\delta>0$ s.t. if $0<x<\delta$ then $f(x)=\frac{1}{x}>M$

Proof Let $M>0$ be arbitrary.

Then let $\delta = \frac{1}{M} > 0$, so

if $0<x<\delta=\frac{1}{M}$ then

$$f(x) = \frac{1}{x} > \frac{1}{\delta} = \frac{1}{\frac{1}{M}} = M$$

☒

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Horizontal Asymptotes

Definition (Formal Definition of Limit at Infinity)

Let f be a function defined on some open interval (a, ∞). Then, we say that

1. *the limit of $f(x)$ as x approaches ∞ is L, and we write $\lim_{x\to\infty} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $M > 0$ such that*

$$\text{if } \ x > M, \quad \text{then } \ |f(x) - L| < \varepsilon.$$

In this case, we say that the line $y = L$ is a horizontal asymptote of $f(x)$ at ∞.

Horizontal Asymptotes

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   $$\text{if } \ x > M, \quad \text{then } \ |f(x) - L| < \varepsilon.$$

   In this case, we say that the line $y = L$ is a horizontal asymptote of $f(x)$ at ∞.

2. *the limit of $f(x)$ as x approaches $-\infty$ is L, and we write $\lim_{x\to-\infty} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $M > 0$ such that*

   $$\text{if } \ x < -M, \quad \text{then } \ |f(x) - L| < \varepsilon.$$

Horizontal Asymptotes

Definition (Formal Definition of Limit at Infinity)

Let f be a function defined on some open interval (a, ∞). The limit of $f(x)$ as x approaches ∞ is L, and we write $\lim_{x\to\infty} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $M > 0$ such that

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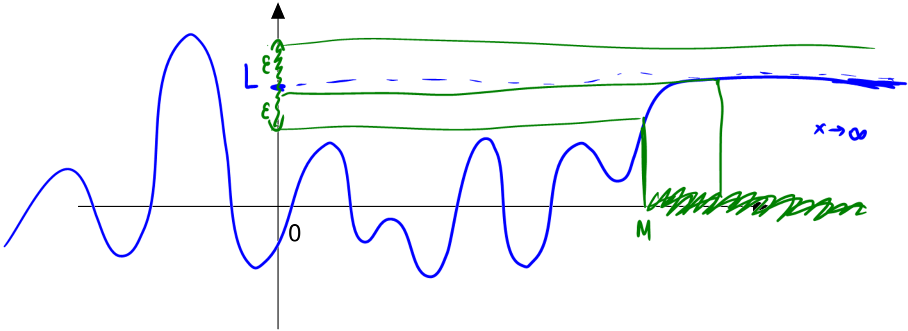


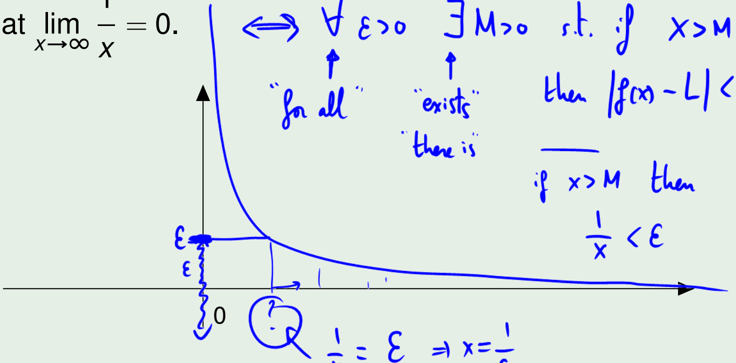
Example

Prove that $\lim_{x\to\infty} \frac{1}{x} = 0.$

$\iff \forall \varepsilon > 0 \ \exists M > 0$ s.t. if $x > M$ then $|f(x) - L| < \varepsilon$

($\forall$ = "for all", $\exists$ = "exists", "there is")

if $x > M$ then $\frac{1}{x} < \varepsilon$



$\frac{1}{x} = \varepsilon \Rightarrow x = \frac{1}{\varepsilon}$

Proof: Let $\varepsilon > 0$ be arbitrary. Let $M = \frac{1}{\varepsilon} > 0$. Then

if $x > M = \frac{1}{\varepsilon}$ then $f(x) = \frac{1}{x} < \frac{1}{M} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$ ∎

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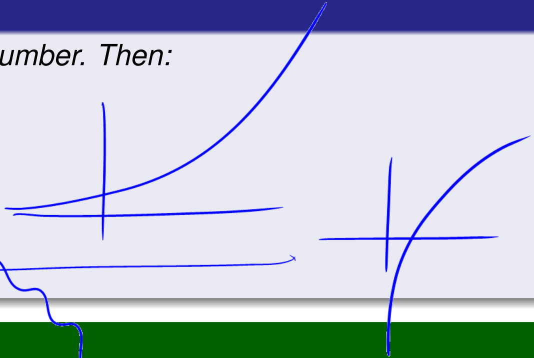
Theorem

Let $r > 0$ be a positive real number. Then:

1. $\lim_{x\to\infty} \frac{1}{x^r} = 0,$
2. $\lim_{x\to\infty} e^x = \infty,$
3. $\lim_{x\to 0^+} \log(x) = -\infty.$



Example

Calculate $\lim_{x\to\infty} e^{-x}$.

(A) ∞

(B) $-\infty$

(C) 0

(D) 1

(E) DNE

$$\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

since $\lim_{x\to\infty} e^x = \infty$.

Suppose $p(x)$ and $q(x)$ are two polynomials. In order to calculate

$$\lim_{x \to \infty} \frac{p(x)}{q(x)}$$

we divide first the numerator and denominator by the highest power of x that appears in the denominator.

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Example

Calculate $\lim_{x\to\infty} \dfrac{3x^3-1}{7x^3+2x-5}$. =

$$= \lim_{x\to\infty} \frac{\frac{3x^3}{x^3} - \frac{1}{x^3}}{\frac{7x^3}{x^3} + \frac{2x}{x^3} - \frac{5}{x^3}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x^3}}{7 + \frac{2}{x^2} - \frac{5}{x^3}}$$

$$= \frac{\lim_{x\to\infty} 3 - \frac{1}{x^3}}{\lim_{x\to\infty} 7 + \frac{2}{x^2} - \frac{5}{x^3}} = \frac{3}{7} = 3/7.$$

Suppose $p(x)$ and $q(x)$ are two polynomials. In order to calculate

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Example

Calculate $\displaystyle\lim_{x\to\infty} \frac{3x^3 - 1}{7x^2 + 2x - 5}$.

$$= \lim_{x\to\infty} \frac{3x - \frac{1}{x^2}}{7 + \frac{2}{x} - \frac{5}{x^2}} = \infty$$

Suppose $p(x)$ and $q(x)$ are two polynomials. In order to calculate

$$\lim_{x\to\infty} \frac{p(x)}{q(x)}$$

we divide first the numerator and denominator by the highest power of x that appears in the denominator.

Example

Calculate $\displaystyle \lim_{x\to\infty} \frac{3x^2 - 1}{7x^3 + 2x - 5}$.

$$= \lim_{x\to\infty} \frac{\frac{3}{x} - \frac{1}{x^3}}{7 + \frac{2}{x^2} - \frac{5}{x^3}} = \frac{0}{7} = 0.$$

The same trick works a little more generally, with algebraic functions.

Example

Calculate $\lim_{x\to\infty} \dfrac{\sqrt{3x^3-1}}{7x^2+2x-5}$.

$$\frac{1}{x^2}\cdot\sqrt{A} = \sqrt{\frac{A}{x^4}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{3}{x}-\frac{1}{x^4}}}{7+\frac{2}{x}-\frac{5}{x^2}} = \frac{\sqrt{\lim_{x\to\infty} \frac{3}{x}-\frac{1}{x^4}}}{\lim_{x\to\infty} 7+\frac{2}{x}-\frac{5}{x^2}} = \frac{0}{7} = 0.$$

.

Beware of $\infty - \infty$ limits! $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$ ... indeterminates

Example

Calculate $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$. $= L$

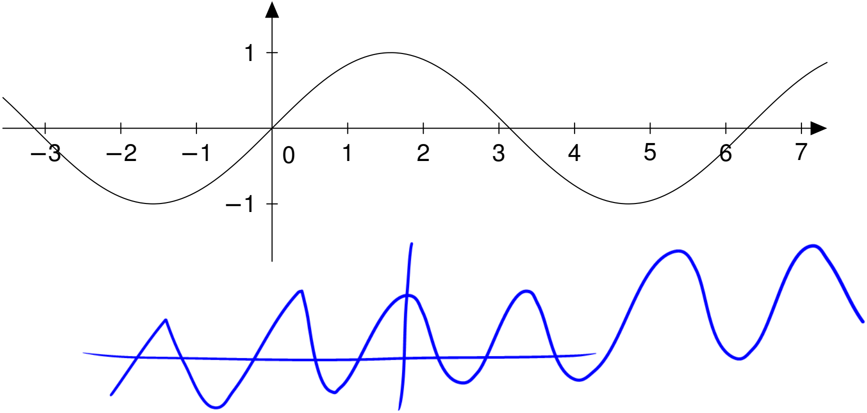
$$\left(L^2 = \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)^2 = \lim_{x\to\infty} x^2+1+x^2-2x\sqrt{x^2+1} = \lim_{x\to\infty} 2x^2+1-2x\sqrt{x^2+1} \quad ☹ \right)$$

$$L = \lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\cdot\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)} = \lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

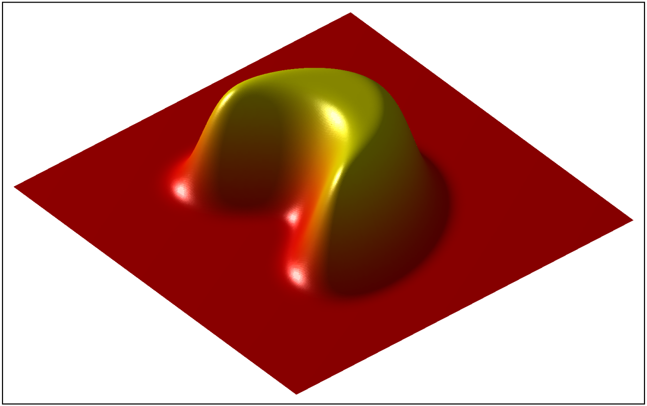
$$= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = 0 \quad (\text{denominator} \to \infty)$$

Continuity

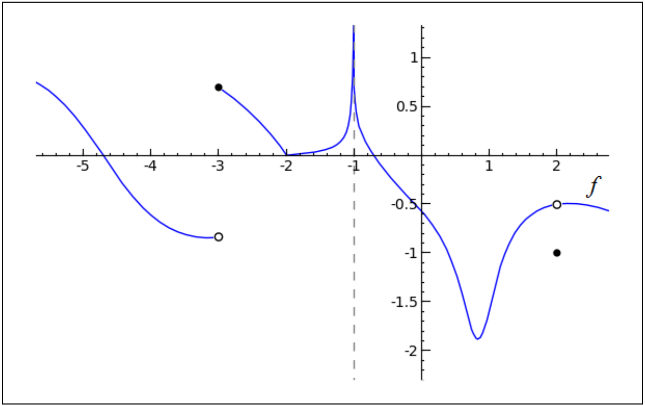
Continuity



Continuity



Dis-Continuity



Dis-Continuity?

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Continuous?

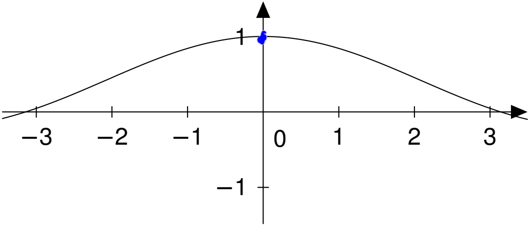
(A) Yes!

(B) No!

(C) No idea!

Dis-Continuity?

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$



Dis-Continuity?

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

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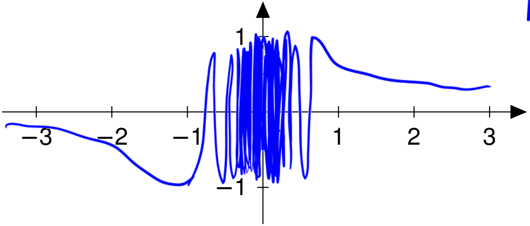
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



Not!

Not at zero.

The Formal Definition of Continuity

Definition

A function $f(x)$ is continuous at a number a if

1. *$f(x)$ is defined at $x = a$, i.e., $f(a)$ is well-defined,*
2. *$\lim_{x \to a} f(x)$ exists, and*
3. *$\lim_{x \to a} f(x) = f(a)$.*

Example

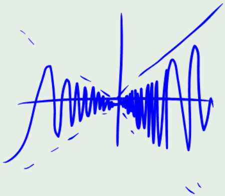
The function $f(x) = x^2$ is continuous at $x = 2$.

- $f(2) = 2^2 = 4$
- $\lim_{x \to 2} x^2 = 4$
- $\lim_{x \to 2} x^2 = 4 = 2^2 = f(2)$ ✓

Example

Is the following function continuous at $x = 0$?

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



- $f(0) = 0$
- $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$

$$-x \leq x \sin\left(\tfrac{1}{x}\right) \leq x$$

As $x \to 0$: $-x \to 0$, $x \to 0$, so by Sq. thm $x\sin\left(\tfrac{1}{x}\right) \to 0$.

- $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 = f(0)$

continuous!

The Formal Definition of Continuity

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2. *$\lim_{x \to a} f(x)$ exists, and*
3. *$\lim_{x \to a} f(x) = f(a)$.*

A function $f(x)$ is continuous from the right at $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a),$$

and the function $f(x)$ is continuous from the left at $x = a$ if

$$\lim_{x \to a^-} f(x) = f(a).$$

Example

The function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1, 1]$.

- it is continuous on $(-1,1)$ which is in the domain.

At $x = 1$: $\lim_{x \to 1^-} 1 - \sqrt{1 - x^2} = 1$ ✓

$\sqrt{1-x^2} \to 0$

$f(1) = 1 - \sqrt{1-1} = 1$ cont. at $x=1$

Similarly at $x = -1$.

.

Theorems about Continuous Functions

Theorem

The following types of functions are continuous at every number in their domain: polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions.

Theorem

If $f(x)$ and $g(x)$ are continuous functions at $x = a$, and c is a constant, then the following functions are also continuous at a:

$$f(x) + g(x),\ f(x) - g(x),\ cf(x),\ f(x)g(x),\ \frac{f(x)}{g(x)} \text{ if } g(a) \neq 0.$$

Theorems about Continuous Functions

Theorem

If f is continuous at $x = b$, and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

Theorems about Continuous Functions

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Example

Calculate $\lim_{x\to 3} \log\left(\frac{x+1}{x-2}\right)$. $= \log\left(\lim_{x\to 3} \frac{x+1}{x-2}\right) = \log 4.$

$\searrow$ 4 and log is cont at 4

ex $\lim_{x\to 0} e^{x\sin\frac{1}{x}} = e^{\lim x\sin\frac{1}{x}} = e^{0} = 1$

Theorems about Continuous Functions

Theorem

If $g(x)$ is continuous at $x = a$, and $f(x)$ is continuous at $g(a)$, then the composite function $f(g(x))$ is continuous at $x = a$.

Theorems about Continuous Functions

Theorem

If $g(x)$ is continuous at $x = a$, and $f(x)$ is continuous at $g(a)$, then the composite function $f(g(x))$ is continuous at $x = a$.

Example

The function $\log(x^2 + 1)$ is continuous in the interval...

x^2+1 is always >0.

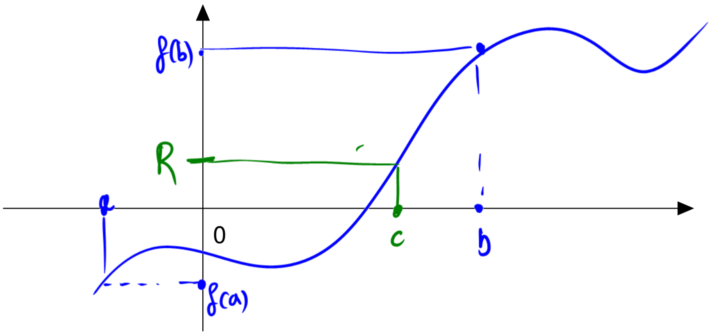
$\Rightarrow \log(x^2+1)$ is always def'd

$\rightarrow$ continuous on $(-\infty, \infty)$.

Theorems about Continuous Functions

Theorem (The Intermediate Value Theorem)

Suppose that $f(x)$ is continuous on the interval $[a, b]$, such that $f(a) \neq f(b)$, and let R be any real number between $f(a)$ and $f(b)$. Then, there is a number c in (a, b) such that $f(c) = R$.



Theorems about Continuous Functions

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Example

Show that the poynomial $f(x) = x^3 + x + 1$ has a root (i.e., a zero value) between -1 and 0.

$f(x)$ is a poly $\Rightarrow$ continuous on $(-\infty, \infty)$

$f(-1) = -1$

$f(0) = 1$

By the inter. value thm: $-1 < 0 < 1$ there is $c \in [-1, 0]$ s.t. $f(c) = 0$

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