Name:

# Final Exam - Practice Questions

**NOTE:** This (mostly) only covers material past the second exam. Please refer to previous practice questions for material from Test 1 and Test 2.

The questions are broken into 3 categories: fundamental questions, advanced questions(*), and challenge questions(**). The fundamental questions test a basic understanding of the definitions and processes. Advanced questions test applying that understanding to more complicated problems. Finally, challenge questions are questions that are intended to stump you. Good luck!

1. Define the following terms:

- Eigenvalue
- Eigenvector
- Eigenspace
- Characteristic polynomial
- Multiplicity of an eigenvalue
- Similar matrices
- Diagonalizable
- Dot product
- Inner product
- Norm (of a vector)
- Orthogonal vectors
- Orthogonal set
- Orthogonal basis
- Orthogonal projection of $\vec{y}$ onto $\vec{u}$
- Unit vector
- Orthonormal set
- Orthonormal basis

2. Find the characteristic equation, eigenvalues, and eigenspaces corresponding to each eigenvalue of the following matrices:

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution:**
$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, $\lambda = 5, -2$, with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$. (The eigenspaces are the span of these eigenvectors).

$\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$, this matrix has complex eigenvalues, so there are no real eigenvalues.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $\lambda_1 = 1, \lambda_2 = 0$, with corresponding eigenspaces $W_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $W_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, $\lambda_1 = 2, \lambda_2 = 3$, with corresponding eigenspaces $W_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and $W_2 = \text{span}\left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda = 1$, with eigenspace all of $\mathbb{R}^3$ (this is the identity matrix, so it times any vector is the same as that vector).

3. Which of the following vectors are eigenvectors of the matrix:

$$\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

(a) $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$

**Solution:** Just check if $A\vec{x} = \lambda\vec{x}$ for some scalar $\lambda$. It turns out only $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector, with eigenvalue 1.

4. Diagonalize the following matrices, if possible:

(a) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ $(\lambda = 1, -2)$ (c) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

**Solution:** (a) is not diagonalizable, the only eigenvalue of the matrix is 2, and the eigenspace corresponding to $\lambda = 2$ is span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, and so there are not 2 linearly independent eigenvectors of this matrix. Therefore there is not a basis for $\mathbb{R}^2$ made of eigenvectors of this matrix, so it is not diagonalizable.
(b) One diagonalization is as follows:

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} = PDP^{-1}$$

where $P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
(c) is not diagonalizable. Similar to part (a), it is impossible to find a basis for $\mathbb{R}^4$ of eigenvectors of this matrix.

5. For each matrix $A$ that was diagonalizable from the previous question, find a formula for $A^k$. That is, find a single matrix whose entries are formulas in terms of $k$ that determines $A^k$.

i.e.

$$\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}^k = \begin{bmatrix} -3 \cdot (-3)^k + 4 \cdot (-2)^k & 6 \cdot (-3)^k - 6 \cdot (-2)^k \\ -2 \cdot (-3)^k + 2 \cdot (-2)^k & 4 \cdot (-3)^k + -3 \cdot (-2)^k \end{bmatrix}$$

**Solution:** So we only need to do this for (b). Using the diagonalization we found:

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} = PDP^{-1}$$

where $P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $P^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

From our work in class see that

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}^k = PD^kP^{-1} = P \begin{bmatrix} (-2)^k & 0 & 0 \\ 0 & (-2)^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} P^{-1}$$

Now computing the product we get:

$$P \begin{bmatrix} (-2)^k & 0 & 0 \\ 0 & (-2)^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} P^{-1} = \begin{bmatrix} -(-2)^k & -(-2)^k & 1 \\ 0 & (-2)^k & -1 \\ (-2)^k & 0 & 1 \end{bmatrix} P^{-1} =$$

$$\begin{bmatrix} 1 & -(-2)^k + 1 & -(-2)^k + 1 \\ (-2)^k - 1 & 2(-2)^k - 1 & (-2)^k - 1 \\ -(-2)^k + 1 & -(-2)^k + 1 & 1 \end{bmatrix}$$

6. Find the eigenvalues of $\begin{bmatrix} 1 & k \\ 2 & 1 \end{bmatrix}$ in terms of $k$. Can you find an eigenvector corresponding to each of the eigenvalues?

**Solution:** Eigenvalues: $\lambda_1 = 1 - \sqrt{2k}$, $\lambda_2 = 1 + \sqrt{2k}$.
Corresponding eigenvectors: $\vec{v}_1 = \begin{bmatrix} -\sqrt{k/2} \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} \sqrt{k/2} \\ 1 \end{bmatrix}$.

7. Rank the following vectors from greatest to least in terms of their norm:

(a) $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 4 \\ 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -1 \end{bmatrix}$

(e) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**Solution:** (c) > (b) > (a) > (d) > (e)

8. For each of the above vectors, find a unit vector that points in the same direction.

**Solution:** (a) $\begin{bmatrix} 1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 4/\sqrt{21} \\ 2/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix}$, (c) $\begin{bmatrix} 2/\sqrt{29} \\ 5/\sqrt{29} \end{bmatrix}$, (d) $\begin{bmatrix} 1/\sqrt{6} \\ 0 \\ -2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \end{bmatrix}$, (e) Does not apply, this vector has no direction.

9. Find a unit vector in $\mathbb{R}^2$ that is orthogonal to $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

**Solution:** We want to find a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $\vec{v} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$. Evaluating this dot product gives the equation $-v_1 + 2v_2 = 0$ so

$$v_1 = 2v_2.$$

Thus, any vector of the form $\vec{u} = v_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Let us take the vector corresponding to $v_2 = 1$, that is, let $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Now we would like a unit vector, but this vector has norm $\sqrt{5}$. Dividing by the norm (length) will yield a unit vector, so our answer is: $\vec{v} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Note that you could also have $\begin{bmatrix} -2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ as a solution.

10. Determine which of the following sets are orthogonal sets:

(a)

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\}$$

(b)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \right\}$$

(c)

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} \right\}$$

**Solution:** (a) is orthogonal, (b) is not orthogonal, and (c) is orthogonal.

11. Find a non-zero vector $\vec{v}$ in $\mathbb{R}^3$ to make the following set an orthogonal set:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \vec{v} \right\}$$

Is the above set (with your selected $\vec{v}$) a basis for $\mathbb{R}^3$? Why does it HAVE to be a basis?

**Solution:** One such vector is $\vec{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ (use techniques similar to question 9). It has to be a basis since an orthogonal set is also linearly independent, and since the dimension of $\mathbb{R}^3$ is 3, any set of three linearly independent vectors must be a basis for $\mathbb{R}^3$ (think about pivots).

12. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$. Calculate $\text{proj}_{\vec{v}}\, \vec{u}$ for the following vectors $\vec{v}$:

(a)
$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

(b)
$$\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

(c)
$$\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 7 \end{bmatrix}$$

**Solution:** (a) $\begin{bmatrix} -21/26 \\ -7/26 \\ -14/13 \end{bmatrix}$, (b) $\begin{bmatrix} 2, 0, -2 \end{bmatrix}$, (c) $\begin{bmatrix} 0 \\ 23/50 \\ -161/50 \end{bmatrix}$

13. Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. Calculate $\text{proj}_{\text{Col}\, A}\, \vec{u}$ for the following matrices $A$:

(a)
$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -1 & -3 \end{bmatrix}$$

**Solution:** (a) This is an orthogonal basis so: $\text{proj}_{\text{Col}\, A}\, \vec{u} = \begin{bmatrix} 12/11 \\ 4/11 \\ 4/11 \end{bmatrix} + \begin{bmatrix} 5/6 \\ -5/3 \\ -5/6 \end{bmatrix} = \begin{bmatrix} 127/66 \\ -43/33 \\ -31/66 \end{bmatrix}$. (b) This is NOT an orthogonal basis, so first we must use the Gramm-Schmidt process to find an orthogonal basis, and then compute the projection onto each basis element and sum them together.

14. For $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, find a vector $\vec{v} \neq \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$ so that $\text{proj}_{\vec{u}}\, \vec{v} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$.

**Solution:** Set up the equation for projection, and similar to problems 9 and 11 before hand, see what this leads to.

15. Find the closest vector to $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ in the subspace

$$W = \text{span} \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

How far is the vector from $\vec{u}$?

**Solution:** The closest vector is: $\begin{bmatrix} -1/5 \\ 0 \\ -2/5 \end{bmatrix}$ which is distance $\sqrt{14/5}$ away from $\vec{u}$.

16. Use the Gramm-Schmidt process to find an orthogonal basis for the column space of the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

**Solution:** $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ -4/3 \\ 3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -25/34 \\ 16/17 \\ 13/34 \\ 9/17 \end{bmatrix} \right\}.$

17. Find the least-squares solution to the following system of equations:

$$\begin{bmatrix} 3 & -1 \\ 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

What is the least-squares error?

**Solution:** The least-squares solution is: $\vec{x} = \left[33/41, 11/82\right]$. The least squares error is: $\frac{\sqrt{1697290}}{574}$ (hah) or about 2.26969