Name:

Test 1 - Practice Questions - Hints and Solutions

1. Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

$$\begin{bmatrix}1&3&5\\2&3&0\\1&0&0\end{bmatrix},\ \begin{bmatrix}1&0&0&0\\0&1&2&0\\0&0&0&1\end{bmatrix},\ \begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix},\ \begin{bmatrix}1&0&2&3\\0&1&0&1\\0&1&2&0\end{bmatrix},\ \begin{bmatrix}1&1&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix},\ \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

Solution: The 2nd, 3rd, and 5th are in row echelon form. The 2nd is the only one in reduced row echelon form.

2. Solve the following system of equations:

$$\begin{array}{rcrcrcr} & & x_2 & + & 5x_3 & = & -4 \\ x_1 & + & 4x_2 & + & 3x_3 & = & -2 \\ 2x_1 & + & 7x_2 & + & x_3 & = & -2 \end{array}$$

Solution: Putting the coefficients into a matrix we obtain the augmented matrix:

$$\begin{bmatrix}0&1&5&-4\\1&4&3&-2\\2&7&1&-2\end{bmatrix}$$

Now we put this matrix into reduced row echelon form and obtain:

$$\begin{bmatrix}1&0&-17&0\\0&1&5&0\\0&0&0&1\end{bmatrix}$$

So since the last row gives the equation $0 = 1$, this system is inconsistent.

3. Solve the following system of equations:

$$\begin{array}{rcrcrcr} 2x_1 & & & - & 6x_3 & = & -8 \\ & & x_2 & + & 2x_3 & = & 3 \\ 3x_1 & + & 6x_2 & - & 2x_3 & = & -4 \end{array}$$

Solution: Putting the coefficients into a matrix we obtain the augmented matrix:

$$\begin{bmatrix}2&0&-6&-8\\0&1&2&3\\3&6&-2&-4\end{bmatrix}$$

Now we put this matrix into reduced row echelon form and obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

So we obtain the solutions $x_1 = 2, x_2 = -1, x_3 = 2$.

4. (a) Is $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ in span$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$? What about $\begin{bmatrix} \pi \\ \log_2 3 \\ 17 \end{bmatrix}$?

Solution: We can form the matrix whose columns are our vectors:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 2 \\ 0 & 3 & 3 \end{bmatrix}$$

and put this matrix into rref:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and since there is a pivot in each row, (i.e. no row of zeros), the vectors span $\mathbb{R}^3$, so both vectors must be in the span.

(b) Is $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$? Is $\begin{bmatrix} \pi \\ \log_2 3 \\ 17 \end{bmatrix}$?

Solution: By the definition of span, these vectors must be linear combinations of those three vectors.

5. Let

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 5 & 0 \\ 0 & 3 & 3 \end{bmatrix}.$$

(a) Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the span of the columns of A? What about $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$?

Solution: If we put A into RREF, we see that there actually is a row of zeros, so we must check these vectors individually. First let's check $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Create the augmented matrix:

$$\begin{bmatrix} 1 & 4 & -1 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

Then put it into RREF to see if there is a solution to this system of equations: We obtain:

$$\begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So this system is consistent, so $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ **IS** in the span.

Now let's check $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Create the augmented matrix:

$$\begin{bmatrix} 1 & 4 & -1 & 3 \\ 1 & 5 & 0 & 2 \\ 0 & 3 & 3 & 1 \end{bmatrix}$$

Then put it into RREF to see if there is a solution to this system of equations:

$$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last row gives the equation 0=1, so this system is inconsistent. Thus, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is **NOT** in the span.

(b) Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination of the columns of A? What about $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$?

Solution: Similar to the previous question, by the definition of span, if a vector is in the span of the columns of A if and only if is a linear combination of the columns of A. Thus, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ **IS** a linear combination of the columns of A, and $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is **NOT** a linear combination of the columns of A

6. Suppose $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(a) Give an example of a vector in span S but not in S.

Solution: Any linear combination of vectors in S is in span S. So for instance we can take $2\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

(b) Give an example of a vector **NOT** in span S.

Solution: If a vector $\vec{v}$ is in span S, then

$$\vec{v} = c\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 2c \\ 0 \\ 3c \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ d \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 2c+d \\ d \\ 3c \end{bmatrix}$$

In particular, notice the 4th entry must be 3 times the 1st entry. So to get a vector not in the span of S, just give an example of a vector in $\mathbb{R}^4$ whose 4th entry is NOT 3 times its 1st entry. For example:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

7. Find a vector $\vec{x}$ such that

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

Solution: This is a matrix equation. To find the solutions, simply solve the augmented matrix:

$$\begin{bmatrix} 2 & 4 & 6 & 2 \\ 4 & 6 & 2 & 6 \\ 6 & 2 & 4 & 4 \end{bmatrix}$$

Putting it into RREF we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

yielding solutions $x_1 = \frac{2}{3}, x_2 = \frac{2}{3}, x_3 = -\frac{1}{3}$. So our vector $\vec{x}$ should be

$$\vec{x} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

8. Calculate the following matrix products if they are defined, otherwise state they are undefined.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 0 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 8 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} =$ product not defined

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 & 5 \\ 7 & 13 & 4 \\ -2 & 15 & -17 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 5 \\ 7 & 13 & 4 \\ -2 & 15 & -17 \end{bmatrix}$

9. (a) Write $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$?

Solution: We wish to solve

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

This is a vector equation which we solve by making the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

and solving it. I leave that part to you. (Put into RREF)

(b) Is the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ linearly independent?

Solution: We need to check if there are any nontrivial solutions to:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We check this by making the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and seeing if there is a free variable. I leave that part to you. (Put into RREF, see if one column pertaining to a variable does not have a pivot). The answer is that there are no free variables, so the set is linearly independent.

(c) Do these vectors span $\mathbb{R}^3$?

Solution: We have a theorem that helps us with this. We form the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and check whether there is a pivot in each row (when in REF), i.e. that there are no rows of zeros. If there are no rows of zeros, then by a theorem we have discussed in class, the columns of this matrix span $\mathbb{R}^3$. Here, the columns of our matrix are exactly the vectors.
The solution is YES they do span $\mathbb{R}^3$.

10. Determine whether the following sets are linearly independent:

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Solution: The idea is to check if $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$ has any non-trivial solutions just like in the problem before. Key things to remember here are that

- if a set contains the zero vector, then the set is linearly dependent
- if a set contains more vectors than the dimension of the vectors (# of entries), then the set is linearly dependent

The answers are: yes, yes, no, no, no, no.

11. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation defined by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+z \\ y+z \end{bmatrix}$$

(a) Show that T is a linear transformation.

Solution: We must check the two properties that define a linear transformation:

- For any $\vec{u}, \vec{v}$, $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$.
- For any $\vec{u}$, c, $T(c\vec{u}) = cT(\vec{u})$.

Let us define arbitrary vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Now simply compute both sides of each equation.

$$T(\vec{u} + \vec{v}) = T\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \right) = \begin{bmatrix} (u_1 + v_1) + (u_3 + v_3) \\ (u_2 + v_2) + (u_3 + v_3) \end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} u_1 + u_3 \\ u_2 + u_3 \end{bmatrix} + \begin{bmatrix} v_1 + v_3 \\ v_2 + v_3 \end{bmatrix} = \begin{bmatrix} u_1 + u_3 + v_1 + v_3 \\ u_2 + u_3 + v_2 + v_3 \end{bmatrix}$$

and by rearranging we see that $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$.
Do the same to check $T(c\vec{u}) = cT(\vec{u})$.

(b) Determine the standard matrix for T.

Solution: To find the standard matrix for T, we must find were T sends the standard basis of the domain of T, in this case $\mathbb{R}^3$.
So, we will calculate:

$$T(\vec{e_1}) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e_2}) = T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T(\vec{e_3}) = T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$$

And now we form the matrix by concatenating these vectors:

$$A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$$

and this matrix A is the standard matrix for T. We can double check that

$$A\vec{x} = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+z\\y+z\end{bmatrix}$$

(c) Is T onto?

Solution: There is a theorem which tells you that T is onto if and only if the columns of the standard matrix of T, that is the matrix A we just found, span the range of T, in this case $\mathbb{R}^2$. So we need to check is the columns of

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$$

span $\mathbb{R}^2$. We have a theorem that says that the columns of a matrix span $\mathbb{R}^n$ precisely when there no row of zeros in RREF, (there is a pivot in every row). So we put A into RREF, which it conveniently already is in, and notice that A has no row of zeros, (it has a pivot in every row). Therefore, the columns of A span $\mathbb{R}^2$, and therefore T is onto.

(d) Is T one-to-one?

Solution: There is a theorem which tells you that T is one-to-one if and only if the columns of the standard matrix of T, that is the matrix A we just found, are linearly independent. So we must check if the set

$$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$$

is linearly independent.

For more detailed steps, see solutions to previous problems on showing sets of vectors are linearly independent.

We form the matrix

$$\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\end{bmatrix}$$

and put it into RREF. Conveniently it already is in RREF, and we see that c_3 is a free variable, and thus this set of vectors is not linearly independent, the set is linearly dependent. Thus, T is not one-to-one.

12. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}2x_1 + x_2\\x_1 - x_2\end{bmatrix}$$

(a) Show that T is a linear transformation.

Solution: See previous problem for idea.

(b) Determine the standard matrix for T.

Solution: See previous problem for idea, the answer is

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

(c) Is T onto?

Solution: See previous problem for idea, the answer is yes.

(d) Is T one-to-one?

Solution: See previous problem for idea, the answer is yes.

13. Determine if the following matrices are invertible and, if so, find the inverse matrix.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 0 & -1 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$

Solution: Following the method we have seen to determine if a matrix is invertible and find the inverse matrix, you can check that (b) and (e) are NOT invertible, and:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 0 & -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 3/4 & -3/4 & -1/4 \\ 1/4 & -1/4 & 1/4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 & 0 \\ 3/4 & -3/4 & -1/4 & 0 \\ 1/4 & -1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$