Chapter 3: Determinants

1. Definition of determinant of a $n \times n$ matrix, in terms of minors:
\[
\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}
\]
2. Calculating determinants as an expansion about any row or column.
3. If $A$ is triangular, then $\det A$ is the product of the entries in the main diagonal.
4. Properties of determinants under elementary row operations:
   - (S) If one row of $A$ is multiplied by $k$ to produce $B$, then $\det B = k \cdot \det A$.
   - (I) If two rows of $A$ are interchanged to produce $B$, then $\det B = - \det A$.
   - (R) If a multiple of one row of $A$ is added to another row to produce $B$, then $\det B = \det A$.
5. A square matrix $A$ is invertible if and only if $\det A \neq 0$.
6. $\det A = \det A^T$, where $A^T$ is the transpose of $A$, also $\det AB = \det A \cdot \det B$.

Chapter 4: Vector Spaces

1. Definition of vector space, and (linear) subspace.
2. Examples of vector spaces: $\mathbb{R}^n$, $P_n(\mathbb{R})$, $M_{m \times n}(\mathbb{R})$, etc.
3. If $v_1, \ldots, v_p$ are vectors in $V$, then $\text{Span}(\{v_1, \ldots, v_p\})$ is a linear subspace of $V$.
4. Definitions of Null subspace $\text{Nul}(A)$, Row space $\text{Row}(A)$, and Column subspace $\text{Col}(A)$ of a matrix.
5. Definition of linear transformation $T: V \to W$ of vector spaces $V,W$:
   - (i) $T(u + v) = T(u) + T(v)$, (ii) $T(\lambda u) = \lambda T(u)$, for all $u, v \in V$ and all $\lambda \in \mathbb{R}$.
6. Definition of Kernel $\text{Ker}T$ and Range (or Image) of a linear transformation $T$.
7. Linear dependence: $S = \{v_1, \ldots, v_n\}$ in a vector space $V$ is a linearly dependent set if there is a linear dependence $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.
8. Definition of basis of a vector space: $B = \{b_1, \ldots, b_p\}$ is a basis of $V$ if (i) $B$ is a linearly independent set, and (ii) $V = \text{Span}(\{b_1, \ldots, b_p\})$. In other words, the vectors in $B$ are a minimal generating set of $V$.
9. If $V = \text{Span}(v_1, \ldots, v_p)$, then some subset of $\{v_1, \ldots, v_p\}$ form a basis of $V$.
10. The pivot columns of a matrix $A$ (i.e., those columns that contain pivots in an echelon form for $A$) form a basis of $\text{Col}(A)$.
11. If $B = \{b_1, \ldots, b_p\}$ is a basis of $V$, then for each $v \in V$ there are unique scalars $\lambda_1, \ldots, \lambda_p$ such that $v = \lambda_1 b_1 + \cdots + \lambda_p b_p$. The (column) vector $[v]_B = (\lambda_1, \ldots, \lambda_p)$ are called the coordinates of $v$ with respect to $B$. The map $V \to \mathbb{R}^p$ that sends $v \mapsto [v]_B$ is called the coordinate mapping, and it is one-to-one and onto.
13. Suppose $B = \{e_1, \ldots, e_n\}$ is the canonical basis of $V$, and $B' = \{b_1, \ldots, b_p\}$ is another basis of $V$. Then, the matrix $P_{B'} = (b_1 \ b_2 \ \cdots \ b_p)$ with columns $b_1, \ldots, b_p$ is called the change-of-coordinates matrix from $B'$ to $B$. It satisfies:
\[
[v]_B = P_{B'} \cdot [v]_{B'} \quad \text{and} \quad [v]_{B'} = P_{B'}^{-1} \cdot [v]_B.
\]
14. If a vector space $V$ has a basis with $n$ elements, then any set of $m > n$ vectors of $V$ must be linearly dependent. Moreover, every basis of $V$ has exactly $n$ elements. In this case, we say that $V$ has dimension $n$.
15. If $H$ is a linear subspace of $V$, then $\dim H \leq \dim V$.
16. If $\dim V = n$, then any $n$ linearly independent vectors of $V$ form a basis of $V$.
17. The dimension of $\text{Nul}(A)$ is the number of free variables in $Ax = 0$, and the dimension of $\text{Col}(A)$ is the number of pivots in an echelon form of $A$.
18. Definition of row space, the space spanned by the rows of a matrix. If two matrices are row equivalent, then their row spaces are identical. The rank of a matrix is the dimension of $\text{Col}(A)$.
19. Rank-Nullity Theorem: $\text{rank } A + \dim \text{Nul}(A) = n$, for any $m \times n$ matrix $A$. 