1. Let $S$ be the set of vectors given below, and let $A$ be the following matrices:

$$S = \begin{bmatrix}
1 & 0 & 0 & 1 \\
2 & 3 & 0 & 2 \\
0 & 0 & 2 & k \\
0 & 0 & 0 & 2k
\end{bmatrix}, \quad A = \begin{bmatrix}
1 & 2 & 0 & 1 & 2 \\
0 & 3 & 0 & 3 & 0 \\
0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 2 & k
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 3 & 0 & 3 \\
0 & 0 & 2 & 1 \\
0 & 0 & 2 & k
\end{bmatrix}$$

where $k$ is a fixed constant. Clearly label the statements as TRUE or FALSE, or as MAYBE if there is not enough information to decide. In all cases, briefly explain your claim.

(a) The matrices $A$ and $B$ have 4 pivot columns each (that is, in echelon form, there are 4 pivots).

**FALSE**

A and B in echelon form have line: \( \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & k-1 \end{bmatrix} \) and \( \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & k-1 \end{bmatrix} \)

respectively. Thus:

If \( k=1 \), there are 3 pivots, and if \( k \neq 1 \), there are 4 pivots.

(b) The determinant of the matrix $B$ in non-zero, therefore $B$ is invertible.

**FALSE**

\[
\begin{align*}
det B &= det \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & k-1 \end{pmatrix} \\
&= 1 \cdot 3 \cdot 2 \cdot (k-1) \quad \text{because the det of upper triangle is product of diagonal entries. Thus det } B \neq 0 \quad \text{if } k \neq 1
\end{align*}
\]

(c) The vector \( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \) belongs to the linear subspace Span($S$).

**FALSE**

We need to solve \( \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & k-1 \end{bmatrix} \) \( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \)

and this system is inconsistent if \( k=1 \), and consistent if \( k \neq 1 \)

(d) The dimension of Nul($A$) is 1.

**FALSE**

Nul($A$) are the solutions of $Ax = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & k-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

and the dimension of Nul($A$) is the number of non-pivot cols. \( = 1 \) if \( k \neq 1 \)

(e) The rank of $A$ and the rank of $B$ coincide.

**TRUE**

\[
\text{rank}(A) = \dim \text{Col}(A) = \dim \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ k \end{bmatrix} \right\} \\
= \dim \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ k \end{bmatrix} \right\} = \dim \text{Col}(B) = \text{rank}(B) \quad \text{bic } \left( \frac{3}{3} \right) = 2 \left( \frac{1}{1} \right)
\]

(f) The dimension of Col($A$) plus the dimension of Nul($A$) is 5.

**TRUE**

\[
\dim \text{Col}(A) = \text{rank}(A) \quad \text{by rank theorem their sum } = \# \text{ columns } = 5
\]

\[
\dim \text{Nul}(A) = \text{nullity}(A)
\]
2. Compute the determinants of the following matrices. **Indicate** what method(s) you use to calculate the determinants.

(a) \( \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \)

\[
\det \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = \boxed{-2}
\]

(b) \( \begin{pmatrix} 2 & 0 & -3 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{pmatrix} \)

**I will expand the determinant along 1st column:**

\[
\det \begin{pmatrix} 2 & 0 & -3 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{pmatrix} = 2 \cdot \det \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 0 & -3 \\ 4 & 5 \end{pmatrix}
\]

\[
= 2 \cdot (-2) - 1 \cdot (0 + 12) = -16
\]

(by part (a))

(c) \( \begin{pmatrix} 2 & 0 & 0 & 0 \\ 7 & 2 & 0 & -3 \\ 1 & 3 & 4 & 2 \\ -\sqrt{2} & 0 & 4 & 5 \end{pmatrix} \)

**I will expand along 1st row:**

\[
\det \begin{pmatrix} 2 & 0 & 0 & 0 \\ 7 & 2 & 0 & -3 \\ 1 & 3 & 4 & 2 \\ -\sqrt{2} & 0 & 4 & 5 \end{pmatrix} = 2 \cdot \det \begin{pmatrix} 7 & 2 & 0 \ 0 & 4 & 5 \ \end{pmatrix} + 0
\]

\[
= 2 \cdot (-16) = \boxed{-32}
\]

(by part (b))

(d) Are the matrices in (a), (b), and (c) invertible? Explain in one line.

**A non-singular matrix is invertible \( \iff \det A \neq 0 \), so (yes)**

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3. (a) Define what it means for a subset $H$ to be a linear subspace of a vector space $V$.

A subset $H \subseteq V$ is a linear subspace if $H$ is also a vector space. We need:
1. $0 \in H$
2. $H$ is closed under $+$ and $-$.

(b) Show that $H = \left\{ \begin{pmatrix} 2a - 3c \\ a + 2b + 3c \\ 4b + 5c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ is a linear subspace of $\mathbb{R}^3$.

Note that

$$H = \left\{ a \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + c \begin{pmatrix} -3 \\ 3 \\ 5 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ 5 \end{pmatrix} \right\}$$

Since Span of a subset is always a linear subspace, we are done.

(c) For $H \subseteq \mathbb{R}^3$ defined as above, find a basis of $H$. Prove that the set you found is a basis.

Let's prove that $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ 5 \end{pmatrix} \right\}$ is a basis of $H$.

By (b), $H = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ 5 \end{pmatrix} \right\}$. It suffices to show now that the vectors are linearly independent, and this is equivalent to the matrix

$$\begin{pmatrix} 2 & 0 & -3 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{pmatrix}$$

being invertible, which is true by 4(d).

(d) For $H$ as in (b), is true that $H = \mathbb{R}^3$? Explain.

$H$ has a basis with 3 elements, so it is 3-dimensional inside $\mathbb{R}^3$, which is also 3-dimensional.

By a theorem in class, this shows that $H = \mathbb{R}^3$. 
4. Let \( H \) be the subset of \( P_2(\mathbb{R}) \) (i.e., the vector space of polynomials of degree \( \leq 2 \)) given by

\[
H = \text{Span} \{ 2 + x, 2x + 4x^2, -3 + 3x + 5x^2 \}).
\]

(a) The set \( H \) is a linear subspace of \( P_2(\mathbb{R}) \). How do you know?

Because it is given as the \( \text{Span} \) of vectors in \( P_2(\mathbb{R}) \) and \( \text{Span} S \) is always a subspace.

(b) Find a basis for \( H \), and prove that it is a basis. What is the dimension of \( H \)?

If we write the vectors in coordinates w.r.t. basis \( \{ 1, x, x^2 \} \)

(via coordinate mapping \( P_2(\mathbb{R}) \rightarrow \mathbb{R}^3 \))

\[
(a+b+x^2) \rightarrow \left( \begin{array}{c} a \\ b \\ c \\ \end{array} \right)
\]

then \( H \) maps to \( \text{Span} \left\{ \left( \begin{array}{c} 2 \\ 0 \\ 0 \\ \end{array} \right), \left( \begin{array}{c} 0 \\ 2 \\ 0 \\ \end{array} \right), \left( \begin{array}{c} -3 \\ 3 \\ 5 \\ \end{array} \right) \right\} \)

We have calculated in 3(c) that this is a basis,

therefore \( H \) is of dimension 3.

(c) Prove that \( H = P_2(\mathbb{R}) \).

By 3(d), we have \( H \cong \mathbb{R}^3 \)

\[ \cong P_2(\mathbb{R}) \]

so \( H \) is 3-dimensional inside \( P_2(\mathbb{R}) \), also 3-dimensional

and by a theorem in class we must have equality, i.e.,

\[ H \cong P_2(\mathbb{R}) \]
5. Let \( B = \begin{bmatrix} (1) & (0) & (0) \\ (0) & (1) & (0) \\ (0) & (0) & (1) \end{bmatrix} \) and \( B' = \begin{bmatrix} (1) & (0) & (1) \\ (0) & (1) & (0) \\ (-1) & (0) & (2) \end{bmatrix} \).

(a) Show that \( B' \) is a basis of \( \mathbb{R}^3 \).

Any basis of \( \mathbb{R}^3 \) consists of 3 linearly independent vectors, so it suffices to show \( B' \) is linearly independent, which is equivalent to \( \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \) being invertible \( \iff \det \neq 0 \). Since \( \det = 1 \cdot 1 \cdot 2 - (-1) \cdot (1) \cdot 1 = 1 \) we conclude this is a basis.

(b) Let \( M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \). Verify that the inverse matrix of \( M \) is \( \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \). It is enough to check that their product is the identity matrix.

\[
\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

(c) Find the change of basis matrix \( P_{B'} \) from \( B' \) to \( B \). Also, find its inverse matrix \( P_{B'}^{-1} \).

\( P_{B'} \) is given by \( \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \), that is, the matrix with \( B' \) as columns.

Its inverse is in part (b): \( \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = P_{B'}^{-1} \)

(d) Let \( v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and \( w = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). Find the coordinates of \( v \) and \( w \) with respect to the basis \( B' \). In other words, find \( [v]_{B'} \) and \( [w]_{B'} \).

\[
[v]_{B'} = P_{B'}^{-1} \cdot [v]_B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}
\]

\[
[w]_{B'} = P_{B'}^{-1} \cdot [w]_B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}
\]

(e) Let \( v = 1 + 2x + 3x^2 \) in \( P_2(\mathbb{R}) \). Find the coordinates of \( v \) with respect to the basis \( \{1 - x^2, x, -1 + 2x^2\} \) of \( P_2(\mathbb{R}) \), and write \( v \) as a linear combination of the basis vectors.

In coordinates, with basis \( \frac{1}{2}, x, x^2 \), this problem is asking to write \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) with basis \( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \), which we have done above, i.e., \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) is \( \begin{bmatrix} 5 \\ 2 \end{bmatrix} \). This means \( 1 + 2x + 3x^2 = 5 \cdot (1 - x^2) + 2 \cdot x + 3 \cdot (-1 + 2x^2) \).
6. Let $A$ be the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 3 \end{pmatrix}$.

(a) Find a basis of the column space of $A$.

$\text{Col } A$ has a basis formed by these vectors in pivot columns:

$\text{Col } A = \text{Span} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(b) Find a basis of the null space of $A$.

$\text{Null } A$ has dimension equal to the # of non-pivot cols = 1 in this case.

$A x = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 7/2 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x = \frac{-1}{2} t, y = \frac{-3}{2} t, z = \frac{7}{2} t, t = t$

Hence $\text{Null } A = \text{Span} \begin{bmatrix} \frac{-1}{2} \\ \frac{-3}{2} \\ \frac{7}{2} \end{bmatrix}$ and \begin{bmatrix} \frac{-1}{2} \\ \frac{-3}{2} \\ \frac{7}{2} \end{bmatrix} is a basis.

(c) Find a basis of the row space of $A$.

$\text{Row } A = \text{Row } (\text{echelon } A) = \text{Span} \begin{bmatrix} 1, 0, 0, \frac{1}{2} \\ 0, 1, 0, -\frac{3}{2} \\ 0, 0, 1, \frac{7}{2} \end{bmatrix}$

Since $\dim \text{Row } A = \dim \text{Col } A = 3 \Rightarrow$ that's a basis of $\text{Row } A$.

(d) State the Rank-Nullity Theorem (also known as the Rank Theorem) in general (for $n \times m$ matrices), and verify that it holds for the matrix $A$ above.

$\dim \text{Col } A + \dim \text{Null } A = \# \text{cols of } A$

$3 + 1 = 4$
7. BONUS:

(a) Let $H_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}$. Show that $H_2$ is a linear subspace of $M_{2\times2}(\mathbb{R})$.

- $(0, 0)$ satisfies $0 + 0 = 0$ so it is in $H_2$.
- If $(a, b), (e, f)$ are in $H_2$, then $a + d = 0 = g + h$ and $(a, b) + (e, f) = (a + e, b + f)$ has $(a + e) + (g + h) = 0 + 0 = 0$.
- If $(a, b)$ is in $H_2$, then $\lambda(a, b) = (\lambda a, \lambda b)$ and $\lambda a + \lambda d = \lambda(a + d) = \lambda \cdot 0 = 0$.

(b) Let $n \geq 1$ be an arbitrary positive integer, and let $H_n$ be the subset of $M_{n\times n}(\mathbb{R})$ formed by those matrices such that the sum of the $n$ elements in the main diagonal adds up to 0 (note that the set $H_2$ is described in part (a)). Show that $H_n$ is a subspace of $M_{n\times n}(\mathbb{R})$.

(c) What is the dimension of the space $H_n$ of part (b)?

(Write the solution on the back of the page.)