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Review text:

Since the 1980's there has been a large amount of literature and interest on elliptic surfaces and Mordell–Weil lattices, due to their many applications in very different settings. The book under review is an introduction to the theory of the Mordell–Weil lattice of an elliptic curve over a function field, and its applications to various problems. The text, authored by two experts in the subject of arithmetic surfaces, is written "with a broad readership in mind." This means that the authors have included introductory chapters on the basic tools needed as prerequisites to Mordell–Weil lattices (e.g., lattices, elliptic curves, algebraic surfaces), as well as more advanced chapters that cover in detail some of the more recent and interesting applications (e.g., finding elliptic curves of high rank over $\mathbb{Q}$, or sphere packings), a combination of topics at different levels that strike the right balance for a specialized textbook. The writing is beautifully clear and the theory is illustrated by many examples, together with historical notes at the end of each chapter. The many pages of references at the end of the book are also a great tool for those that want to dig deeper into research articles on the subject. The one thing sorely missing from this book are exercises at the end of each chapter.

The theory of Mordell–Weil lattices was laid out by Noam Elkies and Tetsuji Shioda in the late 1980's (see for example [1], [2]) and has since grown into a vast body of literature (with a large number of articles in the subject having been contributed by Elkies, Schütt, and Shioda, among many others – we refer the reader to the long list of references at the end of the book). Let us first briefly describe Mordell–Weil lattices and then we will summarize the contents of the book.

Let k be a perfect field, and let $K = k(t)$ be a function field over k in one variable. An elliptic surface $\mathcal{E}$ over the field k can be regarded as an elliptic curve E over the field K, that is, as a smooth projective curve E of genus 1, defined over K, with at least one point defined over K. If we assume that E/K is non-constant (more precisely, non-isotrivial), then the Mordell–Weil–Néron theorem shows that the set of K-rational points on E can be endowed with the structure of a finitely generated abelian group. In particular, $E(K) \cong E(K)_{\mathrm{tors}} \oplus \mathbb{Z}^{R_{E/K}}$, where $E(K)_{\mathrm{tors}}$ is the finite subgroup formed by torsion points (points of finite order) and $R_{E/K} \geq 0$ is a non-negative integer called the rank of the elliptic curve E/K. Using intersection theory, one can define a $\mathbb{Q}$-valued symmetric bilinear pairing $\langle \cdot, \cdot \rangle$ on $E(K)$ (called the height pairing) which induces a positive-definite lattice on the quotient $E(K)/E(K)_{\mathrm{tors}}$. The Mordell–Weil lattice of E/K is precisely the lattice given by the pair $(E(K)/E(K)_{\mathrm{tors}}, \langle \cdot, \cdot \rangle)$.

The contents of the book are organized as follows. The first few chapters contain an introduction to the theories of lattices (with special emphasis on sphere packings, root lattices, and Weyl groups), elliptic curves, algebraic surfaces (highlighting intersection theory, and the Enriques-Koraira classification of surfaces), and elliptic surfaces (restricting their attention early on to jacobian elliptic surfaces). While each one of those topics can be (and are) the subject of a volume of their own, the authors do a good job at summarizing the most important elements that are needed later in the core of the book. The next three chapters introduce the Mordell–Weil lattice, and apply the general results to the study of rational elliptic surfaces, describing in detail their hierarchy which is dominated by the E_8-root lattice. Chapter 9 treats the Galois representations that are naturally associated to the action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ on the Mordell–Weil lattice, and describes the connection between Mordell–Weil lattices, algebraic equations, and the invariant theory of Weyl groups. Chapter 10 describes applications to classical topics, such as the so-called multiplicative excellent families, the 27 lines on a cubic surface, or the 28 bitangents to a plane quartic. Chapters 11 and 12 deal with the lattice theory of elliptic $K3$ surfaces, discuss isogeny structures, and classify all elliptic fibrations on a given $K3$ surface, in particular supersingular $K3$ surfaces. Finally, Chapter 13 discusses applications to finding elliptic curves of high rank over $\mathbb{Q}$ and $\mathbb{Q}(t)$, and applications of Mordell–Weil lattices to constructing high-density sphere packings that were found by Elkies and Shioda in the late 1980's.

References

[1] Elkies, N.D.: *On Mordell-Weil Lattices*, Math. Arbeitstagung, Max-Plank-Institute für Mathematik Bonn (1990).

[2] Shioda, T.: *Mordell–Weil lattices and Galois representations*, I, II, III. Proc. Jpn. Acad. 65A, 268-271; 296-299; 300-303 (1989)