A QUICK INTRO TO ALGEBRAIC GEOMETRY

REFERENCES: SILVERMAN'S "AEC", CHAPTERS 1 & 2

DUMMIT & FOOTE "ABSTRACT ALGEBRA", CHAPTER 15

K a perfect field (i.e., every alg. ext'n is separable)

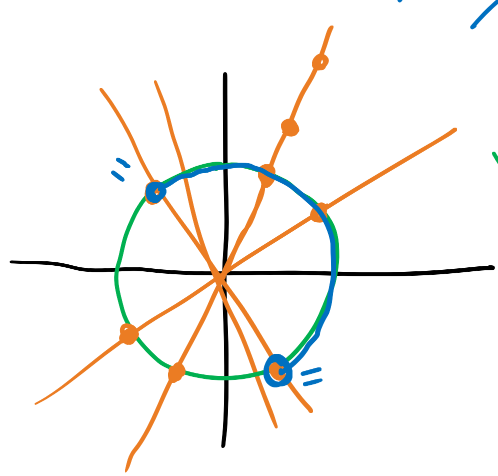
↳ $K = \mathbb{Q}$ or any field of char 0 or any finite field.

$\overline{K}$ = alg. closure of K, $G_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois gp of K (gp of field auto. of $\overline{K}$ that fix K).

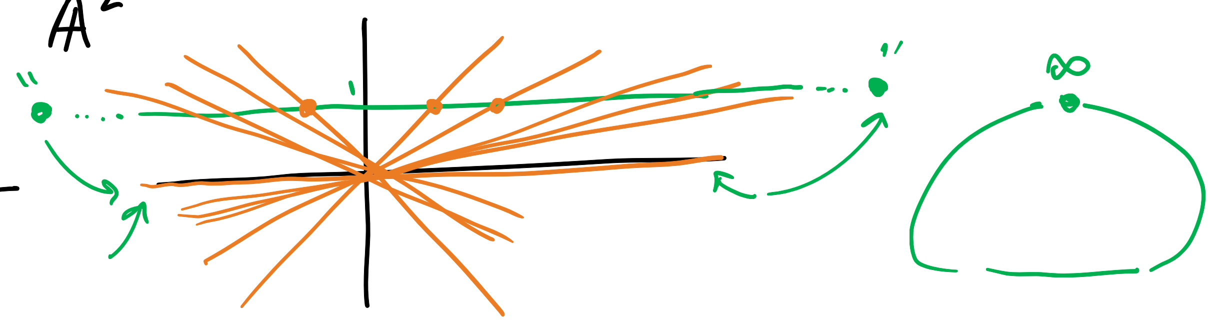
Affine space $\mathbb{A}^n(\overline{K}) = \{(x_1, \dots, x_n) : x_i \in \overline{K}\}$, $\mathbb{A}^n(K) = \{x_i \in K\}$

Proj. space $\mathbb{P}^n(\overline{K}) = \mathbb{A}^{n+1}/\sim$ $\quad \bar{x} \sim \bar{y} \iff \bar{x} = \lambda \cdot \bar{y}$, $\lambda \in \overline{K}^*$

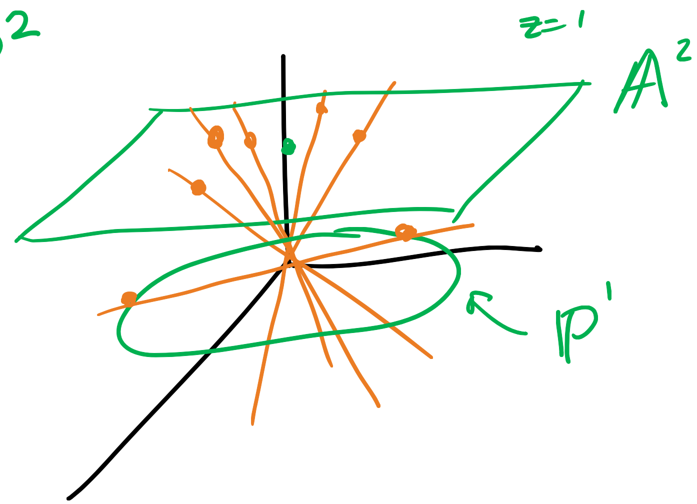
ex $\mathbb{P}^1$

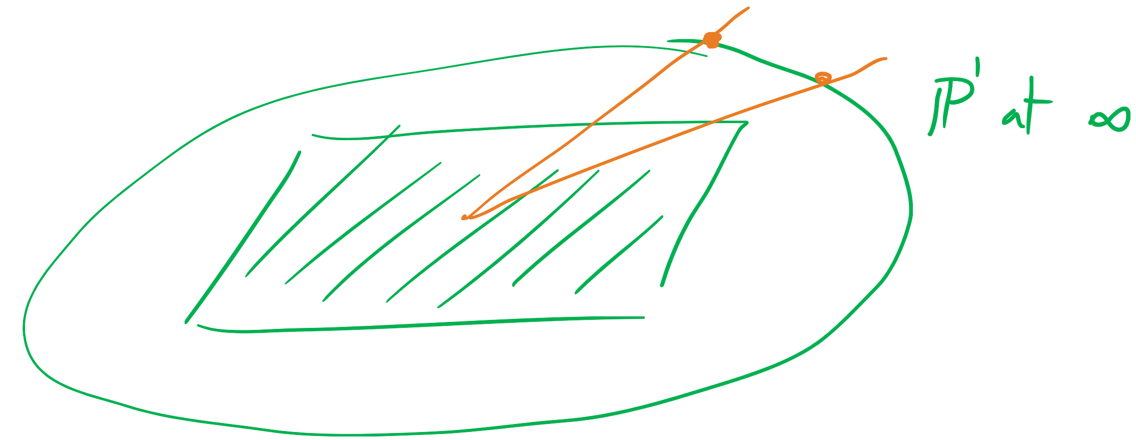


$\mathbb{A}^2$



ex $\mathbb{P}^2$





Note $\mathrm{Gal}(\overline{K}/K)$ acts on $\mathbb{A}^n$ coordinate wise, $\sigma \in G_K$ $\qquad \mathbb{A}^n \circlearrowleft G_K$

$$\sigma \cdot (x_1, \dots, x_n) = (\sigma(x_1), \dots, \sigma(x_n))$$

Def Let $\overline{K}[X] = \overline{K}[x_1, \dots, x_n]$, $I \subseteq \overline{K}[X]$ an ideal

$$V_I = \{P \in \mathbb{A}^n : f(P) = 0 \text{ for all } f \in I\},$$ alg. set assoc. to I.

THM. (Hilbert's basis thm) $I \subseteq \overline{K}[X]$ is fin. generated.

If V is an alg. set. $I(V) = \{f \in \overline{K}[X] : f(P) = 0 \ \forall P \in V\}$
ideal of V

ex $V = \{(x, e^x) : x \in \mathbb{R}\}$

ex $(x) \subseteq \overline{\mathbb{Q}}(x,y)$

$V_{(x)} = \{P \in \mathbb{A}^2 : a = 0\}$, $P = (a,b)$

$= y$-axis.

ex $V = \{(0,0)\}$

$I(V) = (x, y)$

V/K : an algebraic set is said to be <u>defined</u> over K if its ideal $I(V)$ can be generated by polynomials in $K[X]$.

$$I \subseteq \overline{K}[X] \rightsquigarrow V_I$$

<u>ex</u> $(\sqrt{2}\cdot x) \subseteq \overline{\mathbb{Q}}[x,y]$ $\qquad V_I = y\text{-axis}/\mathbb{Q}$

$\quad ||$

$(x) \subseteq \mathbb{Q}[x,y]$