A QUICK INTRO TO ALGEBRAIC GEOMETRY

GOAL: elliptic curves are smooth projective varieties of dimension 1, and genus 1, with at least one rational point.

PREVIOUSLY:

AFFINE

- Affine space: $\mathbb{A}^n(\bar{K}) = \{(x_1, \dots, x_n) : x_i \in \bar{K}\}$
- Proj. space: $\mathbb{P}^n(\bar{K}) = \mathbb{A}^{n+1}(\bar{K})^* / \sim$, $\bar{x} \sim \bar{y} \iff \bar{x} = \lambda \bar{y}$, $\lambda \in \bar{K}^*$.
- Algebraic set: $\bar{K}[X] = \bar{K}[X_1, \dots, X_n]$, $I \subseteq \bar{K}[X]$ an ideal

$$V_I = \{P \in \mathbb{A}^n : f(P) = 0 \quad \forall f \in I\}$$

- Ideal of an alg set V: $I(V) = \{f \in \bar{K}[X] : f(P) = 0 \quad \forall P \in V\}$
- V is defined over K if its ideal $I(V)$ can be generated by polynomials in $K[X]$.

TODAY: Proj. alg. sets, varieties, fn. fields, dimension, maps.

EXAMPLES:

- $I = (x) \subseteq \bar{\mathbb{Q}}[x,y]$ then $V = \{(0,b) : b \in \bar{\mathbb{Q}}\}$ (the y-axis)
- If $V = \{(0,0)\}$ then $I(V) = (x,y)$.

QUESTION: Is $\mathbb{Z} \subseteq \mathbb{A}^1(\bar{\mathbb{Q}})$ an algebraic set?

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No! If it was, $V = \mathbb{Z} \subseteq \mathbb{A}^1(\overline{\mathbb{Q}})$

$$\underbrace{I(V)}_{\text{principal}} \subseteq \underbrace{\overline{\mathbb{Q}}[x]}_{\text{PID}}$$

so $I(V) = (f(x))$ and $n \in \mathbb{Z} = V \Rightarrow f(n) = 0 \ \forall n \in \mathbb{Z}$

$\overset{?}{\Rightarrow} f(x) = 0$.

but $V_{(0)} = \overline{\mathbb{Q}} \neq \mathbb{Z}$ ⨳

DEF: $I \subseteq \overline{k}[X]$ homogeneous

$V_I = \{P \in \mathbb{P}^n(\overline{k}) : f(P) = 0 \ \forall f \in I\}$ is a proj. alg. set.

If V is a proj. alg. set, $I(V) = \{f \in \overline{k}[X] : f$ is homog. and $f(P) = 0 \ \forall P \in V\}$

EXAMPLES:

- $I = (X) \subseteq \overline{\mathbb{Q}}[X, Y, Z]$

$V_I = \{[0, b, c] : b, c \in \overline{\mathbb{Q}}$ not both zero!$\} \subseteq \mathbb{P}^2$

$\cup$

$[0,1,0] = \mathcal{O}$ is in all "vertical" lines

Fix $a_0 \in \overline{\mathbb{Q}}$

- $I = (X - a_0 Z) \subseteq \overline{\mathbb{Q}}[X, Y, Z]$

$V_I = \{[a_0, b, 1] : b \in \overline{\mathbb{Q}}\}$

$\cup \{[0,1,0]\}$

"$\mathcal{O} \in V_I$"

- $I = (X, Y) \subseteq \overline{\mathbb{Q}}[X, Y, Z]$

$V_I = \{[0,0,1]\} \subseteq \mathbb{P}^2$

FROM AFFINE TO PROJECTIVE AND BACK

(Affine) $I = (y^2 - x^3 + x) \subseteq \mathbb{Q}[x,y]$

$V_I = \{(a,b) \in \mathbb{Q}^2 : b^2 = a^3 - a\} \subseteq \mathbb{A}^2(\mathbb{Q})$

$V_I : y^2 = x^3 - x$

HOMOGENIZE $\longrightarrow$ PROJECTIVE CLOSURE

$C : f(x,y) = 0 \rightsquigarrow f^*(X,Y,Z) = Z^{\deg f} \cdot f\left(\frac{X}{Z}, \frac{Y}{Z}\right)$

$f(x,y) = y^2 - x^3 + x \rightsquigarrow f^*(X,Y,Z) = Z^3 \cdot \left(\left(\frac{Y}{Z}\right)^2 - \left(\frac{X}{Z}\right)^3 + \frac{X}{Z}\right)$

$= ZY^2 - X^3 + Z^2 \cdot X \stackrel{?}{=} 0$

(proj closure) $I = (ZY^2 - X^3 + Z^2 \cdot X)$

$V_I^{proj} = \{[a,b,c] \in \mathbb{P}^2 : f^*(a,b,c) = 0\} \subseteq \mathbb{P}^2$

Affine charts of $\mathbb{P}^2 = \{[a,b,c] : \dots\}$

$\rightsquigarrow Z=1 \quad \{[a,b,1]\} \subseteq \mathbb{P}^2 \longleftrightarrow \{(a,b)\} \subseteq \mathbb{A}^2$, $\{[a,b,1]\} \cong \mathbb{A}^2$

$\rightsquigarrow Y=1 \quad \{[a,1,c]\} \cong \mathbb{A}^2$

$\rightsquigarrow X=1 \quad \{[1,b,c]\} \cong \mathbb{A}^2$

Affine chart of V is $V \cap$ "$\mathbb{A}^n$"

$V: \; zy^2 = x^3 - z^2 x$ dehomogenize.

$$V \cap \text{“}\mathbb{A}^2\text{”}_{\{z=1\}} = \{[x, y, 1] : \underline{y^2 = x^3 - x}\}$$

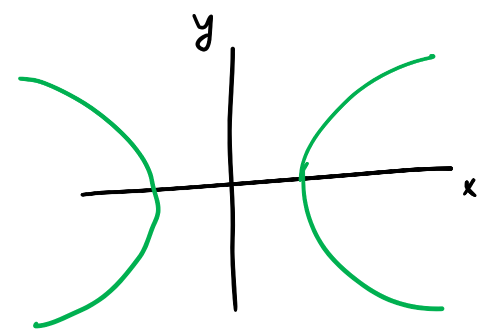
$$V \cap \text{“}\mathbb{A}^2\text{”}_{\{x=1\}} = \{[1, y, z] : zy^2 = 1 - z^2\}$$

or $zy^2 + z^2 = 1$

ex

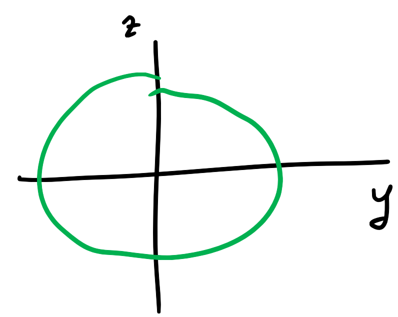
$V: \; x^2 - y^2 = z^2$

$$V \cap \text{“}\mathbb{A}^2\text{”}_{\{z=1\}} : \; x^2 - y^2 = 1$$



$$V \cap \text{“}\mathbb{A}^2\text{”}_{\{x=1\}} : \; 1 - y^2 = z^2$$

or $1 = y^2 + z^2$



DEF: A (projective) algebraic set V is called a (projective) **variety** if its (homog.) ideal is a prime ideal in $\overline{K}[X]$.

(EQUIV: V is a variety if it's irreducible:
$$V \neq V_1 \cup V_2 \text{ of two proper alg. sets})$$

NOTE! Check primality in $\overline{\overline{K}}[X]$

DEF: Let V/K be a affine variety. Then:

- $K[V] = K[X] / I(V)$ – affine coordinate ring.
- $K(V) =$ field of fractions of $K[V]$ – function field of V

If V/K is a proj. variety, then
$$K[V] := K[V \cap \text{"}A^n\text{"}], \quad K(V) = K(V \cap \text{"}A^n\text{"})$$

EXAMPLES:

- $(x) \subseteq \overline{\mathbb{Q}}[x,y]$ prime ideal!
- $(x,y) \subseteq \overline{\mathbb{Q}}[x,y]$ prime ideal!
- $I = (x \cdot y) \subseteq \overline{\mathbb{Q}}[x,y]$ NOT PRIME

  $V_I = \{(0,b) : b \in \overline{\mathbb{Q}}\} \cup \{(a,0) : a \in \overline{\mathbb{Q}}\}$
- $(x^2+y^2) \subseteq \overline{\mathbb{Q}}[x,y]$ NOT PRIME
- $(\underbrace{y^2-x^3+x}_{\text{irred}}) \subseteq \underbrace{\overline{\mathbb{Q}}[x,y]}_{\text{UFD}}$ prime ideal!

EXAMPLES:

- $I = (x) \subseteq \mathbb{Q}[x,y]$ $\qquad \mathbb{Q}[V] = \dfrac{\mathbb{Q}[x,y]}{(x)} \cong \mathbb{Q}[y]$

  V_I = y-axis $\qquad \mathbb{Q}(V) = \mathbb{Q}(y)$

- $I = (y^2 - x^3 + x) \subseteq \mathbb{Q}[x,y]$ $\qquad \mathbb{Q}[V] = \dfrac{\mathbb{Q}[x,y]}{(y^2-x^3+x)}$

  V_I : $y^2 = x^3 - x$ $\qquad \mathbb{Q}(V) = \mathbb{Q}(x, \sqrt{x^3-x})$

DEF: Let V/K be a variety. The <u>dimension</u> of V is the transcendence degree of $\overline{K}(V)$ over $\overline{K}$.

ex V : $y^2 = x^3 - x$, $\overline{\mathbb{Q}}(V) = \overline{\mathbb{Q}}(x, \sqrt{x^3-x})$

$$\underbrace{\overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}(x)}_{\text{trans. deg} = 1} \underbrace{\subseteq \overline{\mathbb{Q}}(x, \sqrt{x^3-x})}_{\text{alg. ext'n.}} \Rightarrow \dim V = 1$$

ex V : y-axis, $I = (x) \subseteq \mathbb{Q}[x,y]$

$$\overline{\mathbb{Q}}(V) \cong \underbrace{\overline{\mathbb{Q}}(y) / \overline{\mathbb{Q}}}_{\text{trans. deg} = 1}$$

$\Rightarrow \dim V = 1.$

DEF: A curve is a proj. variety of dimension 1.

DEF: Let C be a curve defined by $f(X,Y,Z)=0$.

Then C is singular at P iff $\left.\frac{\partial f}{\partial x}\right|_p = \left.\frac{\partial f}{\partial y}\right|_p = \left.\frac{\partial f}{\partial z}\right|_p = 0$.

EQUIV: Let $M_p = \{ f \in \bar{K}[V] : f(P)=0 \}$

V variety. $P \in V$ is non-singular $\iff \dim_{\bar{K}} M_p / M_p^2 = \dim V$.

ex $V: ZY^2 = X^3 - Z^2X$

$V: f(X,Y,Z) = 0 = ZY^2 - X^3 + Z^2X$

$$\begin{cases} f_X = -3X^2 + Z^2 = 0 \\ f_Y = 2ZY = 0 \\ f_Z = Y^2 + 2ZX = 0 \end{cases}$$

$f_Y = 0 \Rightarrow \begin{cases} Z=0,\ X=0,\ Y=1 \quad P=[0,1,0]\ f_Z(P) \neq 0 \\ Z=1,\ Y=0,\ f_Z=0 \Rightarrow X=0,\ f_X=0 \Rightarrow Z=0 \quad P \neq [0,0,0]\ X \end{cases}$

no singular points on V. (smooth!)

ex $V: y = x^3$

Singular as a proj. curve!

DEF: Let V be a variety, and $P \in V$.

The <u>local ring</u> of V at P, $\overline{K}[V]_P$, is the localization of $\overline{K}[V]$ at M_P:

$$\overline{K}[V]_P = \left\{ F \in \overline{K}(V) : F = \frac{f}{g},\ f, g \in \overline{K}[V],\ g(P) \neq 0 \right\}$$

we say that F is "regular at P"

DEF: Let V_1, V_2 be proj. vars. A <u>rational map</u> between V_1 and V_2 is a map

$$\phi : V_1 \longrightarrow V_2$$
$$P \longmapsto [f_0(P), \dots, f_n(P)] \in V_2$$

$\neq 0$ $\neq 0$

where $f_0, \dots, f_n \in \overline{K}(V_1)$.

or a pole at 0

DEF: A rational map $\phi : V_1 \to V_2$ is regular at $P \in V_1$ if $\exists\, g \in \overline{K}(V_1)$ s.t.

(i) each $g \cdot f_i$ is regular at P

(ii) for some i we have $(g \cdot f_i)(P) \neq 0$.

If ϕ is regular at all $P \in V_1$ then we say that ϕ is a morphism.

ex $V_1 : 3y^2 = x^3 + 1 \;\cong\; V_2 : y^2 = x^3 + 1$

quadratic twist of V_2

$\phi : V_1 \to V_2$

$(x, y) \mapsto (x, \sqrt{3}\,y)$

$\phi / \mathbb{Q}(\sqrt{3})$

DEF: $V_1 \cong V_2$ are isomorphic if there are morphisms $\phi : V_1 \to V_2$, $\psi : V_2 \to V_1$ s.t. $\phi \circ \psi$, $\psi \circ \phi$ are the identity on V_2, V_1 resp.

ex

$\phi : V_1 \to V_2$

$V_1 = \{ zy^2 = x^3 + z^3 \}$

$V_2 = \mathbb{P}^1$

$[x, y, z] \xmapsto{\phi} [x, z]$

$[0, 1, 0] \longmapsto [1, 0]$

Check ϕ is regular at $[0,1,0]$:

$$[x, z] = [x^3, zx^2]$$
$$= [zy^2 - z^3, zx^2]$$
$$= \underbrace{[y^2 - z^2, x^2]}_{\text{evaluated at } [0,1,0]}$$

get $[1, 0]$ ✓