

A QUICK INTRO TO ALGEBRAIC GEOMETRY

GOAL: elliptic curves are smooth projective varieties of dimension 1, and genus 1, with at least one rational point.

PREVIOUSLY:

- Affine space: $A^n(\bar{K}) = \{(x_1, \dots, x_n) : x_i \in \bar{K}\}$
- Proj. space: $IP^n(\bar{K}) = A^{n+1}(\bar{K})^* / \sim$, $\bar{x} \sim \bar{y} \iff \bar{x} = \lambda \bar{y}$, $\lambda \in \bar{K}^*$.

- AFFINE**
- Algebraic set: $\bar{K}[X] = \bar{K}[x_1, \dots, x_n]$, $I \subseteq \bar{K}[X]$ an ideal
 $V_I = \{P \in A^n : f(P) = 0 \ \forall f \in I\}$
 - Ideal of an alg set V : $I(V) = \{f \in \bar{K}[X] : f(P) = 0 \ \forall P \in V\}$
 - V is defined over K if its ideal $I(V)$ can be generated by polynomials in $K[X]$.

TODAY: Proj. alg. sets, varieties, fn. fields, dimension, maps.

EXAMPLES:

- $I = (x) \subseteq \bar{\mathbb{Q}}[x, y]$
then $V = \{(0, b) : b \in \bar{\mathbb{Q}}\}$
(the y-axis)
- If $V = \{(0, 0)\}$
then $I(V) = (x, y)$.

QUESTION:

Is $\mathbb{Z} \subseteq A^1(\bar{\mathbb{Q}})$
an algebraic set?

QUESTION: Is $Z \subseteq \mathbb{A}^1(\overline{\mathbb{Q}})$ an algebraic set?

No! If it was algebraic, $V = Z$

then $I(V) \subseteq \overline{\mathbb{Q}}[x]$
is principal PID

$I(V) = (f(x))$, and $n \in Z \in V$

$\Rightarrow \underline{f(n) = 0} \quad \forall n \in Z \Rightarrow f(x) = 0.$

$\Rightarrow I(V) = (0)$

but $V_{(0)} = \overline{\mathbb{Q}} \neq Z \quad \times$

DEF: Let $I \subseteq \overline{\mathbb{K}}[x]$ homogeneous ideal

$V_I = \{ P \in \mathbb{P}^n : f(P) = 0 \quad \forall f \in I \}$ is a proj. alg. set.

V is a proj. alg. set, $I(V) = \{ f \in \overline{\mathbb{K}}[x]$

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