A QUICK INTRO TO A.G. (CONTINUES)

PREVIOUSLY...

- Affine and projective space
- $\underbrace{\text{Affine and projective algebraic}}$ sets and their ideals.
    - varieties (when ideal is prime)
- Coordinate ring and function field of a variety
- Dimension (transcendence degree of $\overline{K}(V)/\overline{K}$)
- Curve: proj. variety of dimension 1.
- $C: f(x,y,z)=0$ is singular at $P\in C \iff \left.\frac{\partial f}{\partial x}\right|_P = \left.\frac{\partial f}{\partial y}\right|_P = \left.\frac{\partial f}{\partial z}\right|_P = 0$.
- Local ring of V at P

$$\overline{K}[V] = \overline{k}[X] / I(V)$$

$$\overline{K}(V) = \text{fraction field of } \overline{K}[V]$$

ex

$V = \mathbb{P}^1$

$\mathbb{Q}[x]$

$\mathbb{Q}(x)$

$$\overline{K}[V]_P = \Big\{ F \in \overline{K}(V) : \underbrace{F = \frac{f}{g}}_{\text{"F is regular at P"}},\ f,g \in \overline{K}[V],\ g(P)\neq 0 \Big\}$$

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- Rational map: $\phi : V_1 \longrightarrow V_2$
$$P \longmapsto [f_0(P), \dots, f_n(P)] \in V_2$$
where $f_i \in \overline{K}(V_1)$.

- A rational map ϕ is regular at P if $\exists\, g \in \overline{K}(V_1)$ s.t.
  (i) each $g \cdot f_i$ is regular at P
  (ii) for some i we have $(g \cdot f_i)(P) \neq 0$.

ex $\phi : \{ zy^2 = x^3 + z^3 \} \longrightarrow \mathbb{P}^1$ is regular at $P = [0,1,0]$
$$[x, y, z] \longmapsto [x, z]$$
Take $g = x^2$. Then
$$[x, z] = [x^3, zx^2]$$
$$= [zy^2 - z^3, zx^2]$$
$$= [y^2 - z^2, x^2]$$
evaluated at P is $[1, 0]$ ✓

- If ϕ is regular at all $P \in V_1$, we call it a morphism.

TODAY: ALGEBRAIC CURVES.

Let C be a curve (proj. variety of dimension 1) and let P be a smooth point.

$M_P = \{ f \in \overline{K}[C]_P : f(P) = 0 \}$

DEF: • $\mathrm{ord}_P : \overline{K}[C]_P \longrightarrow \mathbb{Z}^{\geq 0} \cup \{\infty\}$

$$\mathrm{ord}_P(f) = \sup\{ d \in \mathbb{Z} : f \in M_P^d \}$$

extend by $\mathrm{ord}_P(f/g) = \mathrm{ord}_P(f) - \mathrm{ord}_P(g)$ to $\overline{K}(C)$.

$\overline{K}[C]_P$ is a DVR.

• A uniformizer at P is a fn. $t \in \overline{K}(C)$ s.t. $\mathrm{ord}_P(t) = 1$.

PROP. Let $f \in \overline{K}(C)$, then there are finitely many $\underbrace{\text{zeroes}}_{\mathrm{ord}_P(f)>0}$ and $\underbrace{\text{poles}}_{\mathrm{ord}_P(f)<0}$ of f.

If f has NO poles, then $f \in \overline{K}$. (C is smooth) ex $E/\bar{K}$: $y^2 = x^3 + Ax + B = (x-e_1)(x-e_2)(x-e_3)$ $e_i \in K$.

Smooth!
(e_1, e_2, e_3 are all distinct)

Q: $P_i = (e_i, 0)$, $\operatorname{ord}_{P_i}(x-e_i)$?

$O = [0,1,0]$, $\operatorname{ord}_O(x-e_i)$?

$$E : ZY^2 = \underbrace{(X-e_1Z)(X-e_2Z)(X-e_3Z)}_{f(X,Z)}$$

$\bar{K}[E] = \bar{K}[E \cap \text{"}\mathbb{A}^2\text{"}] = \bar{K}[x,y] / (y^2 - x^3 - Ax - B)$, $\{z=1\}$ ‖ "$\mathbb{A}^2$"

$X - e_1Z \rightsquigarrow x - e_1$

- $P_1 = [e_1, 0, 1]$, $X - e_1Z, Y \in M_{P_1}$

Moreover $\langle x-e_1, y \rangle$ is maximal in $\bar{K}[E] \Rightarrow$ they gen. M_{P_1}.

Taylor expansion around $x = e_1$

get: $(e_1-e_2)(e_1-e_3)(X-e_1Z)Z^2 = ZY^2 - (X-e_1Z)^3 - (2e_1 - e_2 - e_3)(X-e_1Z)^2 Z$. $(*)$

Since Z does not vanish at P_1, we see that $X - e_1Z \in M_{P_1}^2$.

Then $M_{P_1}/M_{P_1}^2$ is generated by Y and $\dim_{\bar{K}} M_{P_1}/M_{P_1}^2 = 1$ b/c smooth!

$\Rightarrow$ $\boxed{\operatorname{ord}_{P_1} Y = 1.}$

From $(*)$ we deduce $\operatorname{ord}_{P_1}(X-e_1Z) = \min\{2\operatorname{ord}_{P_1} Y, \cancel{2\operatorname{ord}_{P_1}(X-e_1Z)}, \cancel{3\operatorname{ord}_{P_1}(X-e_1Z)}\}$

Note $\operatorname{ord}_{P_1}(X-e_1Z) > 0$, then $\boxed{\operatorname{ord}_{P_1}(X-e_1Z) = 2\operatorname{ord}_{P_1} Y = 2.}$

- Similarly $\operatorname{ord}_{p_i}(x-e_i z)=2$, $\operatorname{ord}_{p_i}(Y)=1$. for $i=1,2,3$.
- Let $\mathcal{O}=[0,\underline{\underline{1}},0]$. The ideal $M_{\mathcal{O}}$ is generated by x, z

$$x=\frac{X}{Y} \qquad z=\frac{Z}{Y} \qquad \overline{K}[E]=\overline{K}[E\cap \text{"}A^2\text{"}_{\{Y=1\}}]=\overline{K}[x,z]/(\qquad)$$

x, z generate $M_{\mathcal{O}}$

$\rightarrow$ $X-e_iZ \in M_{\mathcal{O}}$ (for $i=1,2,3$) and $ZY^2=(X-e_1Z)(X-e_2Z)(X-e_3Z)$

$\Rightarrow$ $z\in M_{\mathcal{O}}^3$. and $Y\notin M_{\mathcal{O}}$

$\Rightarrow$ $M_{\mathcal{O}}/M_{\mathcal{O}}^2$ is generated just by x $\Longrightarrow$ $\operatorname{ord}_{\mathcal{O}}(x)=1$. $\rightarrow \operatorname{ord}_{\mathcal{O}}(z)=3$.

$$\operatorname{ord}_{\mathcal{O}}(x-e_iz)=\min\{\operatorname{ord}_{\mathcal{O}}(X),\operatorname{ord}_{\mathcal{O}}(z)\}=\min\{1,3\}=1.$$

- Hence $\operatorname{ord}_{\mathcal{O}}(x-e_i)=\operatorname{ord}_{\mathcal{O}}\left(\frac{X-e_iZ}{Z}\right)=1-3=-2.$ pole of order 2.

$\operatorname{ord}_{\mathcal{O}}(y)=\operatorname{ord}_{\mathcal{O}}(Y/Z)=0-3=-3.$ pole of order 3.

PROP. C/K curve, $t \in K(C)$ uniformizer at smooth pt $P \in C$. Then $K(C)$ is a finite separable extension of $K(t)$.

§2. Maps.

PROP. C curve, $V \subseteq \mathbb{P}^n$ variety, $P \in C$ is a smooth point, and $\phi : C \to V$ is a rational map, then ϕ is regular at P. In particular, if C is smooth, then ϕ is a morphism.

THM. $\phi : C_1 \to C_2$ a morphism of curves, then ϕ is either constant or surjective.

- $\phi : C_1/K \to C_2/K$, ϕ/K non-constant, C_1, C_2 smooth.

<u>Def</u>.
$$\left.\begin{aligned} \phi^* : K(C_2) &\longrightarrow K(C_1) \\ f &\longmapsto \phi^*(f) = f(\phi) \end{aligned}\right\} \text{injective.}$$

$$K(C_2) \text{ ``}\subseteq\text{''} K(C_1)$$

THM. Let $C_1, C_2/K$ be curves, and $\phi : C_1 \to C_2$ non-constant.

(a) $K(C_1)/\phi^*(K(C_2))$ is a finite extension.

$\phi^* : K(C_2) \to K(C_1)$

$f \mapsto f \circ \phi$

(b) Let $\tau : K(C_2) \to K(C_1)$ be an injection.

Then there is $\phi : C_1 \to C_2$ s.t. $\phi^* = \tau$.

(c) Let $F \subseteq K(C_1)$ a subfield w/ $[K(C_1) : F] < \infty$ and $K \subseteq F$.

Then there is a curve C'/K smooth, unique up to isom., and a non-constant map $\phi : C_1 \to C'/K$ s.t. $\phi^*(K(C')) = F$.

DEF: $\deg(\phi : C_1 \to C_2) = [K(C_1) : \phi^*(K(C_2))]$

$\deg\phi = [\mathbb{Q}(x, \sqrt{x^3+1}) : \mathbb{Q}(x)]$

$= 2$

ex $\phi : \{zy^2 = x^3 + z^3\}/\mathbb{Q} \to \mathbb{P}^1(\mathbb{Q})$

$[x, y, z] \longmapsto [x, z]$

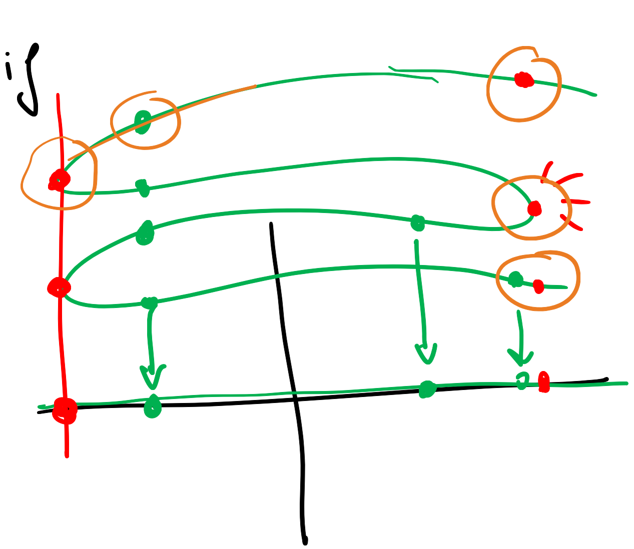
$\phi^* : \mathbb{Q}(x) \to \mathbb{Q}(x, \sqrt{x^3+1})$

$x_{\mathbb{P}^1} \longmapsto x_E$

DEF: ϕ is separable, inseparable, purely inseparable if

$$K(C_1)/\phi^*(K(C_2)) \text{ is .}$$

$$(\deg \phi,\ \deg_s \phi,\ \deg_i \phi)$$



RAMIFICATION:

DEF: Let $\phi : C_1 \to C_2$ non-ct. map of smooth curves, $P \in C_1$.

We define the ramification index of ϕ at P: $\quad e_\phi(P) := \mathrm{ord}_P\left(\phi^*(t_{\phi(P)})\right)$

where $t_{\phi(P)}$ is a uniformizer for C_2 at $\phi(P)$.

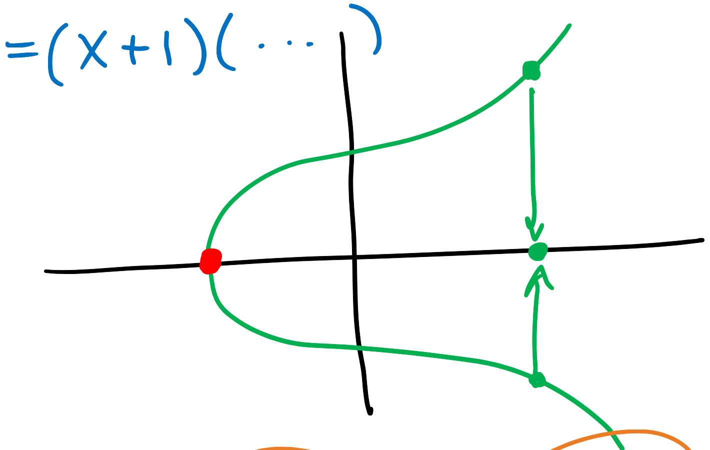
$\left(\mathrm{ord}_{\phi(P)}(t_{\phi(P)}) = 1\right)$

We say that ϕ is unramified at P if $e_\phi(P) = 1$, ramified otherwise.

ex

$$\phi: E \longrightarrow \mathbb{P}^1 \qquad E: zy^2 = x^3 + z^3 \longleftrightarrow y^2 = x^3 + 1$$

$$[x, y, z] \longmapsto [x, z] \qquad = (x+1)(\cdots)$$



- Ramification at $P = [-1, 0, 1]$

$$\phi(P) = [-1, 1]$$

uniformizer $t_{\phi(P)} = x_{\mathbb{P}^1} + 1$

$$\phi^* : \mathbb{Q}(x_{\mathbb{P}^1}) \longrightarrow \mathbb{Q}(E) = \text{fr. } \mathbb{Q}[x,y]/(\quad)$$

$$e_\phi(P) = \mathrm{ord}_P(\phi^*(t_{\phi(P)}))$$

$$\phi^*(x_{\mathbb{P}^1} + 1) = x_E + 1$$

then $\mathrm{ord}_P(x_E + 1) = 2 \implies e_\phi(P) = \mathrm{ord}_P(\phi^*(t_{\phi(P)})) = 2.$