A QUICK INTRO TO ALG. GEOM. (MORE!)

DEF: An elliptic curve is a smooth projective variety of dimension 1, genus 1, and with at least one rational point over the field of definition.

PREVIOUSLY: C curve, P smooth.

- $\mathrm{ord}_P : \overline{K}[C]_P \longrightarrow \mathbb{Z}^{\geq 0} \cup \{\infty\}$, $\mathrm{ord}_P(f) = \sup\{d \in \mathbb{Z} : f \in M_P^d\}$
- $t \in \overline{K}(C)$ is a uniformizer if $\mathrm{ord}_P(t) = 1$
- PROP: $f \in \overline{K}(C)$ then fin. many zeros and poles. No poles $\Rightarrow f \in \overline{K}$.
- ex $E/K : y^2 = (x-e_1)(x-e_2)(x-e_3)$ smooth, $P_i = (e_i, 0)$, $\mathcal{O} = [0,1,0]$. Then
$$\mathrm{ord}_{P_i}(x-e_i) = 2, \quad \mathrm{ord}_{\mathcal{O}}(x-e_i) = -2, \quad \mathrm{ord}_{P_i}(y) = 1, \quad \mathrm{ord}_{\mathcal{O}}(y) = -3.$$
- PROP: $\overline{K}(C)$ is a finite separable ext'n of $K(t)$, where t is a uniformizer at a smooth pt. P.
- $\underset{\text{curve}}{\phi : C} \longrightarrow \underset{\text{variety}}{V} \subseteq \mathbb{P}^n$ with C smooth at P, then ϕ is regular at P.
- $\phi : C_1 \longrightarrow C_2$ morphism of curves. Then ϕ is either constant or surjective (over $\overline{K}$!)

• THM. Let $\phi : C_1/_K \to C_2/_K$ be a non-constant morphism (of curves)

Let $\phi^* : K(C_2) \longrightarrow K(C_1)$, $\phi^*(f) = f(\phi)$. Then:

(0) ϕ^* is injective

(a) $K(C_1)/\phi^*(K(C_2))$ is a finite extension

(b) Let $\tau : K(C_2) \to K(C_1)$ be an injection, then $\exists!\ \phi : C_1 \to C_2$ s.t. $\phi^* = \tau$.

(c) Let $F \subseteq K(C_1)$ be a subfield with $K(C_1)/F$ finite and $K \subseteq F$. Then $\exists\ C'/K$ smooth, unique up to iso., and a non-constant map $\phi : C_1 \to C'/K$ s.t. $\phi^*(K(C')) = F$.

• DEF: $\deg(\phi : C_1 \to C_2) = [K(C_1) : \phi^*(K(C_2))]$

• COR: If $\phi : C_1 \to C_2$ a map of degree 1, C_1, C_2 smooth $\Rightarrow$ ϕ is an isomorphism.

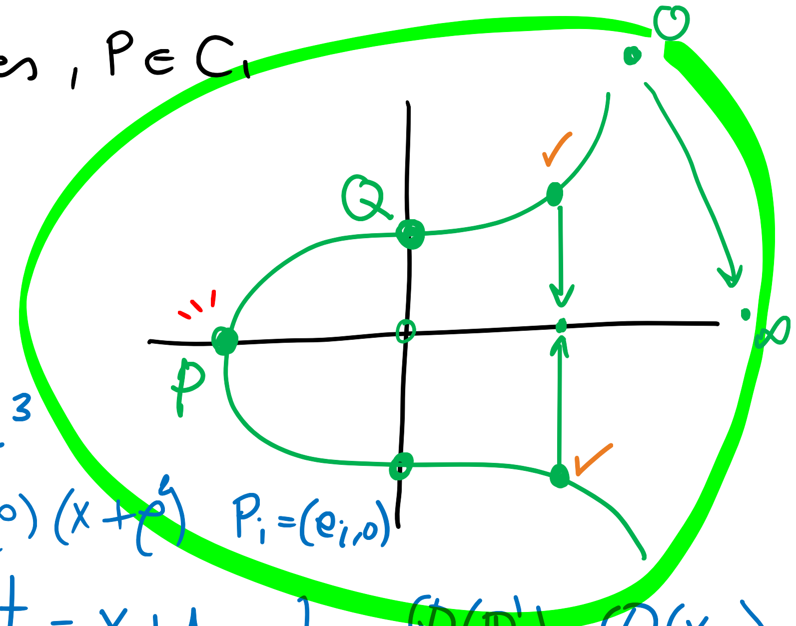
• ex $\phi : \{zy^2 = x^3 + z^3\} \longrightarrow \mathbb{P}^1$ is a map of degree 2. Here

$[x, y, z] \longmapsto [x, z]$

$\phi^*(K(C_2)) = \mathbb{Q}(x) \subseteq \mathbb{Q}(E)$

$\mathbb{Q}(E) =$ frac. field of $\dfrac{\mathbb{Q}[x,y]}{(y^2 - x^3 + 1)}$

• RAMIFICATION: $\phi : C_1 \to C_2$ non-ct., smooth curves, $P \in C_1$



$$e_\phi(P) := \mathrm{ord}_P(\phi^*(t_{\phi(P)}))$$

where $t_{\phi(P)}$ is a uniformizer at $\phi(P)$.

EXAMPLE: $\phi : E \longrightarrow \mathbb{P}^1$ where $E: ZY^2 = X^3 + Z^3$

$[X,Y,Z] \longmapsto [X,Z]$ $\qquad y^2 = (x+1)(x+\rho)(x+\rho^2)$ $\quad P_i = (e_i, 0)$

• $\underline{\text{At } P = [-1,0,1]}$ $\phi(P) = [-1,1]$, a unif. for $\mathbb{P}^1$ at $x=-1$ is $t = x_{\mathbb{P}^1} + 1$ where $\mathbb{Q}(\mathbb{P}^1) = \mathbb{Q}(x_{\mathbb{P}^1})$

Moreover $\phi^* : \mathbb{Q}(x_{\mathbb{P}^1}) \hookrightarrow \mathbb{Q}(x_E, y_E)$ so $\phi^*(x_{\mathbb{P}^1} + 1) = x_E + 1$

$f \longmapsto f(\phi)$

Since $\mathrm{ord}_P(x_E + 1) = 2 \implies e_\phi(P) = \mathrm{ord}_P(\phi^*(t_{\phi(P)})) = 2$. (RAMIFIED!)

(• $\underline{\text{At } Q = [0,1,1]}$ $\phi(Q) = [0,1]$, $t_{\phi(Q)} = x$, $\phi^*(x_{\mathbb{P}^1}) = x_E$

Thus $e_\phi(Q) = \mathrm{ord}_Q(\phi^*(t_{\phi(Q)})) = 1$.

(UNRAMIFIED!)

[NOTE: $y^2 - 1 = x^3$ so

$(y+1)(y-1) = x^3$ so

$M_Q = \langle x, y-1 \rangle = \langle x \rangle$ so

smooth pt $\quad \mathrm{ord}_Q(x) = 1$]

$$\boxed{e_\phi(P) = \mathrm{ord}_P\left(\phi^*(t_{\phi(P)})\right)}.$$

$\phi : \{ZY^2 = X^3 + Z^3\} \longrightarrow \mathbb{P}^1$

$[X, Y, Z] \longmapsto [X, Z]$

- At $\mathcal{O} = [0,1,0]$

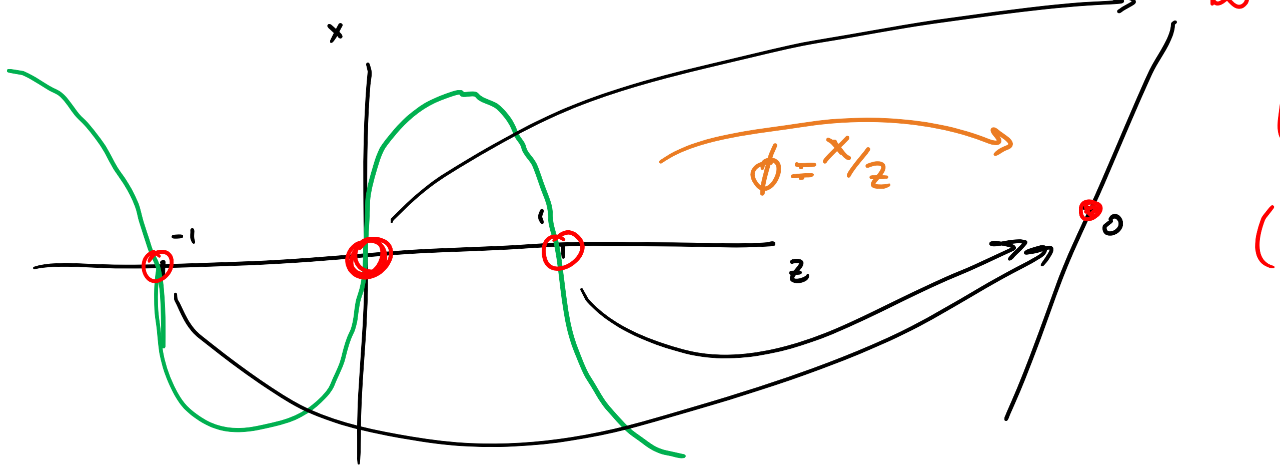
$\phi(\mathcal{O}) = [1,0]$, uniformizer for $\mathbb{P}^1$ at $[1,0]$ is $t_{\phi(\mathcal{O})} = \dfrac{1}{X_{\mathbb{P}^1}}$.

PREVIOUSLY COMPUTED

Thus $\phi^*\left(\dfrac{1}{X_{\mathbb{P}^1}}\right) = \dfrac{1}{X_E}$ and $\mathrm{ord}_{\mathcal{O}}(x) = \mathrm{ord}_{\mathcal{O}}(x+1) = -2$

Hence $\mathrm{ord}_{\mathcal{O}}\left(\dfrac{1}{X_E}\right) = 2$ and $e_\phi(\mathcal{O}) = \mathrm{ord}_{\mathcal{O}}\left(\phi^*(t_{\phi(\mathcal{O})})\right) = 2$. (RAMIFIED!)

NOTE: At chart $\{Y=1\}$ $E : z = x^3 + z^3$

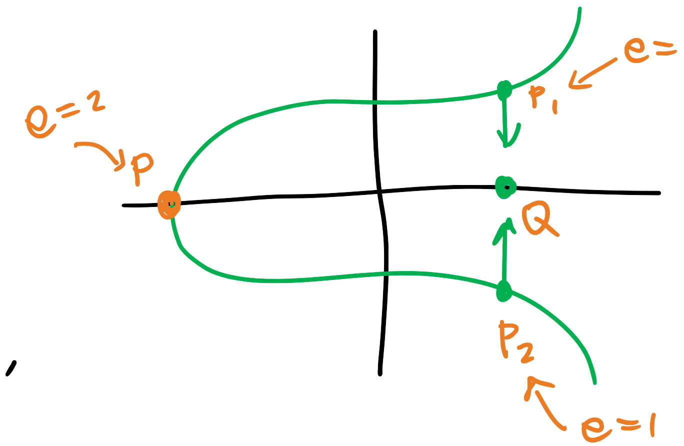


$(x,z) = (0,0)$ is a double pt over ∞

$(1,0), (-1,0)$ are above 0

PROP. $\phi : C_1 \to C_2$ non-ct. smooth curves, $P \in C_1$, $Q \in C_2$

(a) $$\sum_{P \in \phi^{-1}(Q)} e_\phi(P) = \deg \phi.$$



(b) For all but finitely many $Q \in C_2$,

$$\#\phi^{-1}(Q) = \deg_s \phi$$

($\deg_s \phi$ = sep. deg)

(c) $\psi : C_2 \to C_3$ another non-ct.

$$e_{\psi \circ \phi}(P) = e_\phi(P) \cdot e_\psi(\phi(P))$$

COR: $\phi : C_1 \to C_2$ is unramified $\iff$ $\#\phi^{-1}(Q) = \deg_s(\phi)$ $\forall Q \in C_2$.

ex K, $\mathcal{O}_K = \mathbb{Z}[\alpha]$, $f(\alpha) = 0$ $\to$ $\mathcal{O}_K = \dfrac{\mathbb{Z}[x]}{(f(x))}$

p ramifies $\iff$ $f(x)$ mod p repeated roots

$(x-a)^e (x-b)$

DIVISORS. C/K curve

- $\mathrm{Div}(C)$ = free abelian group gen. by points on C.

$$= \left\{ D = \sum_{P\in C} n_P \cdot (P) \ , \ n_P \in \mathbb{Z} \ , \ n_P = 0 \text{ for almost all } P \in C \right\}$$

- $\deg D = \sum_{P\in C} n_P$

- $\mathrm{Div}^0(C) = \{ D \in \mathrm{Div}(C) : \deg D = 0 \}$

- $\mathrm{Gal}(\overline{K}/K) = G_K \circlearrowright \mathrm{Div}(C)$ $\qquad D^\sigma = \sum_{P\in C} n_P \cdot (P^\sigma)$

$\sigma \in$

$P^\sigma = \sigma(P)$ coordinate-wise.

- $\mathrm{Div}_K(C) = \{ D \in \mathrm{Div}(C) : D^\sigma = D \ \ \forall \sigma \in G_K \}$

- $D \in \mathrm{Div}(C)$ is <u>principal</u> if $D = \mathrm{div}(f)$ for some $f \in \overline{K}(C)^*$

where $\boxed{\mathrm{div}(f) = \sum_{P\in C} \mathrm{ord}_P(f) \cdot (P)}$

Recall! $f \in \overline{K}(C)^*$
$\Rightarrow$ only fin. many zeros and poles.

<u>ex</u> $E: y^2 = (x-e_1)(x-e_2)(x-e_3)$
$P_i = (e_i, 0)$

$\mathrm{div}(x-e_1)$
$= 2\cdot(P_1) - 2\cdot(\mathcal{O})$.

NOTE: $\deg(\mathrm{div}(x-e_1)) = 0$

$\mathrm{div}(y) = (P_1) + (P_2) + (P_3) - 3\cdot(\mathcal{O})$.

$\deg(\mathrm{div}(y)) = 0$.

- $\mathrm{Princ}(C) \subseteq \mathrm{Div}(C)$ subgp of princ. divisors $(\mathrm{div}(f))$
- We say $D_1 \sim D_2$ linearly equivalent $\Longleftrightarrow$ $D_1 - D_2$ is principal

  $D_1 = D_2 + \mathrm{div}(f)$ for some $f \in \overline{K}(C)^*$.
- Picard group

$$\mathrm{Pic}(C) = \mathrm{Div}(C) \big/ \mathrm{Princ}(C)$$

Similarly $\mathrm{Pic}_K(C) = \mathrm{Pic}(C)^{G_K}$

- **PROP.** C smooth curve, $f \in \overline{K}(C)^*$

  (a) $\mathrm{div}(f) = 0 \Longleftrightarrow f \in \overline{K}^*$

  (b) $\deg(\mathrm{div}(f)) = 0$ <u>ex</u> $\mathrm{div}(y) = (P_1) + (P_2) + (P_3) - 3\cdot(O)$.

- $\mathrm{Div}^0(C) \supseteq \mathrm{Princ}(C)$, $\mathrm{Pic}^0(C) = \mathrm{Div}^0(C) \big/ \mathrm{Princ}(C)$, $\mathrm{Pic}^0_K(C) = \mathrm{Pic}^0(C)^{G_K}$

- $\phi : C_1 \longrightarrow C_2$, $\phi^* : \mathrm{Div}(C_2) \longrightarrow \mathrm{Div}(C_1)$

$$(Q) \longmapsto \underbrace{\sum_{P \in \phi^{-1}(Q)}}_{\text{RAMIFICATION DIVISOR.}} e_\phi(P) \cdot (P)$$

§. DIFFERENTIALS.

DEF. (Merom.) differentials on C (a curve) is the set Ω_C, which is the $\overline{K}(C)$-vector space generated by <u>symbols</u> of the form dx, for some $x \in \overline{K}(C)$, subject to:

(i) $d(x+y) = dx + dy \quad \forall x, y \in \overline{K}(C)$

(ii) $d(xy) = x\,dy + y\,dx$

(iii) $d\alpha = 0 \quad \forall \alpha \in \overline{K}$.

PROP. 4.2 C curve.

(a) Ω_C is a 1-dim'l $\overline{K}(C)$-vector space

(b) Let $x \in \overline{K}(C)$. Then dx is a $\overline{K}(C)$-basis.

$\iff$ $\overline{K}(C)/\overline{K}(x)$ is finite separable.

RECALL: we saw t is a unif. at P $\to \overline{K}(C)/\overline{K}(t)$ is fin. sep.

PROP. 4.3. $P \in C$, $t \in \overline{K}(C)$ is a uniformizer at P (smooth)

(a) $\forall \omega \in \Omega_C$ $\exists!$ $g \in \overline{K}(C)$ s.t. $\omega = g \cdot dt$

WE CALL $g = $ "$\frac{\omega}{dt}$"

(b) $f \in \overline{K}(C)$ regular at $P \Rightarrow \frac{df}{dt_P}$ is also regular at P.

(c) $\mathrm{ord}_P(\omega) := \mathrm{ord}_P\left(\frac{\omega}{dt_P}\right)$ is well-defined.

(d)

(e) $\tilde{\forall} P \in C$, $\mathrm{ord}_P(\omega) = 0$.

Q

$\deg(\mathrm{div}(\omega)) = 0$??

"NOOO!"

$\frac{\omega}{dt}$ changes w/ P!

$t = t_P$

DEF. $\mathrm{div}(\omega) = \sum_{P \in C} \mathrm{ord}_P(\omega) \cdot (P)$, for $\omega \in \Omega_C$.

ω is regular (or holomorphic) if $\mathrm{ord}_P(\omega) \geq 0$ $\forall P \in C$.

non-vanishing if $\mathrm{ord}_P(\omega) \leq 0$ $\forall P \in C$.

ex $\mathbb{P}^1$, $\overline{K}(\mathbb{P}^1) = \overline{K}(x)$, $\Omega_{\mathbb{P}^1} = \langle dx \rangle$

Compute $\mathrm{div}(dx)$

- If $P \in \mathbb{P}^1$, $P = [x_0, y_0]$, $\begin{cases} P \neq [1,0] & t_P = x - x_0 \quad dt_P = d(x - x_0) \\ & = dx - dx_0 \\ & = dx \Rightarrow \frac{dx}{dt} = 1 \\ P = [1,0] & t_P = \frac{1}{x} \quad dt_P = d\left(\frac{1}{x}\right) = -\frac{dx}{x^2} \end{cases}$

$\mathrm{ord}_P \frac{dx}{dt} = 0$

$\mathrm{ord}_{x=0} -\frac{1}{x^2} = -2$

$$\mathrm{div}(dx) = \sum_{P \in \mathbb{P}^1} \mathrm{ord}_P(dx) \cdot (P)$$

$$= \sum_{\substack{P \in \mathbb{P}^1 \\ P \neq [1,0]}} \mathrm{ord}_P(dx)(P) + \mathrm{ord}_{[1,0]}(dx) \cdot ([1,0])$$

$$= \sum_P 0 \cdot (P) + -2 \cdot ([1,0]) = -2 \cdot ([1,0])$$

$$\Rightarrow \deg \mathrm{div}(dx) = -2.$$

• **DEF.** The canonical divisor class on C is the class of $\mathrm{div}(\omega) \in \mathrm{Pic}(C)$ for any non-zero differential on Ω_C.

ex $\quad y^2 = (x-e_1)(x-e_2)(x-e_3) \;/\overline{K}$. Compute $\mathrm{div}(dx)$?

$$\xrightarrow{d} \quad 2y\,dy = \big((x-e_1)(x-e_2) + (x-e_2)(x-e_3) + (x-e_1)(x-e_3)\big)\,dx$$

$$\Rightarrow dx = \frac{2y\,dy}{\sum_{i\neq j}(x-e_i)(x-e_j)}$$ and y is a uniformizer at P_i, $i=1,2,3$.

$$\Rightarrow \mathrm{ord}_{P_i} dx = \mathrm{ord}_{P_i}\left(\frac{2y}{\sum(x-e_i)(x-e_j)}\right) = 1$$

Also $\frac{x}{y}$ is a unif at $\mathcal{O}$, $\quad d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

$$\rightsquigarrow dx = \frac{2y^3}{2y^2 - x\sum(x-e_i)(x-e_j)}\, d\left(\frac{x}{y}\right)$$

$$\mathrm{ord}_{\mathcal{O}} dx = -9 + 6 = -3$$

$$\Rightarrow \mathrm{div}\, dx = (P_1) + (P_2) + (P_3) - 3(\mathcal{O})$$

$$\Rightarrow \mathrm{div}\,\frac{dx}{y} = 0$$

$\boxed{\frac{dx}{y} \text{ is a non-vanishing holomorphic diff}}$