

A QUICK INTRO TO ALG. GEOM. (MORE!)

DEF: An elliptic curve is a smooth proj. variety of dimension 1, genus 1, and with at least one rational point over the field of definition.

PREVIOUSLY: C curve, P smooth.

- $\text{ord}_P : \overline{K}[C]_P \rightarrow \mathbb{Z} \cup \{\infty\}$, $\text{ord}_P(f) = \sup \{d \in \mathbb{Z} : f \in M_P^d\}$
- $t \in \overline{K}(C)$ is a uniformizer if $\text{ord}_P(t) = 1$
- PROP: $f \in \overline{K}(C)$ then fin. many zeros and poles. No poles $\Rightarrow f \in \overline{K}$.
- ex $E_{/K}$: $y^2 = (x-e_1)(x-e_2)(x-e_3)$ smooth, $P_i = (e_i, 0)$, $\mathcal{O} = [0, 1, 0]$. Then
 $\text{ord}_{P_i}(x-e_i) = 2$, $\text{ord}_{\mathcal{O}}(x-e_i) = -2$, $\text{ord}_{P_i}(y) = 1$, $\text{ord}_{\mathcal{O}}(y) = -3$.
- $K(C)$ is a finite separable ext'n of $K(t)$, where t is a uniformizer at a smooth pt. P .
- $\phi: C \rightarrow V \subseteq \mathbb{P}^n$ with C smooth at P , then ϕ is regular at P .
curve variety
- $\phi: C_1 \rightarrow C_2$ morphism of curves. Then ϕ is either constant or surjective (over \overline{K} !)

• **THM.** Let $\phi: C_1/K \rightarrow C_2/K$ be a non-constant morphism
(of curves)

Let $\phi^*: K(C_2) \rightarrow K(C_1)$, $\phi^*(f) = f(\phi)$. Then:

(a) ϕ^* is injective

(1) $K(C_1)/\phi^*(K(C_2))$ is a finite extension

(2) Let $\tau: K(C_2) \rightarrow K(C_1)$ be an injection, then $\exists! \phi: C_1 \rightarrow C_2$ s.t. $\phi^* = \tau$.

(3) Let $F \subseteq K(C_1)$ be a subfield with $K(C_1)/F$ finite and $K \subseteq F$. Then

$\exists C'/K$ smooth, unique up to iso., and a non-constant map $\phi: C_1 \rightarrow C'/K$
s.t. $\phi^*(K(C')) = F$.

• **DEF:** $\deg(\phi: C_1 \rightarrow C_2) = [K(C_1) : \phi^*(K(C_2))]$

• **COR:** $\exists f \phi: C_1 \rightarrow C_2$ a map of degree 1, C_1, C_2 s

