PREVIOUSLY...

Let C be a curve.

= NEW HW SET = POSTED!

- $\mathrm{Div}(C) = \{ D = \sum_{P \in C} n_P \cdot (P) \; : \; n_P \in \mathbb{Z},\ n_P = 0 \text{ for almost all } P \in C \}$
- $\deg D = \sum_{P \in C} n_P$
- $\mathrm{Div}^0(C) = \{ D \in \mathrm{Div}(C) \; : \; \deg D = 0 \}$
- $f \in \overline{K}(C)^*$, $\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P(f) \cdot (P)$

$\to$ $\mathrm{div}(f) = 0 \iff f \in \overline{K}^*$

$\to$ $\deg(\mathrm{div}(f)) = 0$

$y^2 = (x-e_1)(x-e_2)(x-e_3)$

$\hookrightarrow$ ex $\mathrm{div}(x-e_i) = 2\cdot(P_i) - 2(\mathcal{O})$

$\mathrm{div}(y) = (P_1) + (P_2) + (P_3) - 3\cdot(\mathcal{O})$

- $\mathrm{Princ}(C) = \{ D \; : \; D = \mathrm{div}(f),\ f \in \overline{K}(C)^* \}$
- $\mathrm{Pic}(C) = \mathrm{Div}(C) / \mathrm{Princ}(C)$, $\mathrm{Pic}^0(C) = \mathrm{Div}^0(C) / \mathrm{Princ}(C)$
- $\Omega_C = \{ dx \; : \; x \in \overline{K}(C) \} / \underline{d(x+y) = dx + dy,\ dxy = x\,dy + y\,dx,\ da = 0 \text{ if } a \in \overline{K}}$.

$\hookrightarrow$ Ω_C is a 1-dim'l space over $\overline{K}(C)$

$\hookrightarrow$ if t is a unif. at a smooth pt $P \in C$ $\Rightarrow$ dt is a basis of $\Omega_C / \overline{K}(C)$.

- $\mathrm{div}(\omega) := \sum_{P \in C} \mathrm{ord}_P(\omega)\,(P) = \sum_{P \in C} \mathrm{ord}_P\left(\frac{\omega}{dt_P}\right) \cdot (P)$

PREVIOUSLY...

- ex For $\mathbb{P}^1$, $\Omega_{\mathbb{P}^1} = \langle dx \rangle$, $\operatorname{div}(dx) = -2 \cdot ([1,0])$ ($[1,0] = \infty$)
- ex For $E: y^2 = (x-e_1)(x-e_2)(x-e_3)$, $\Omega_E = \langle \frac{dx}{y} \rangle$, $\underbrace{\operatorname{div}\left(\frac{dx}{y}\right) = 0}$.

  we say dx/y is a non-vanishing, holom. differential.

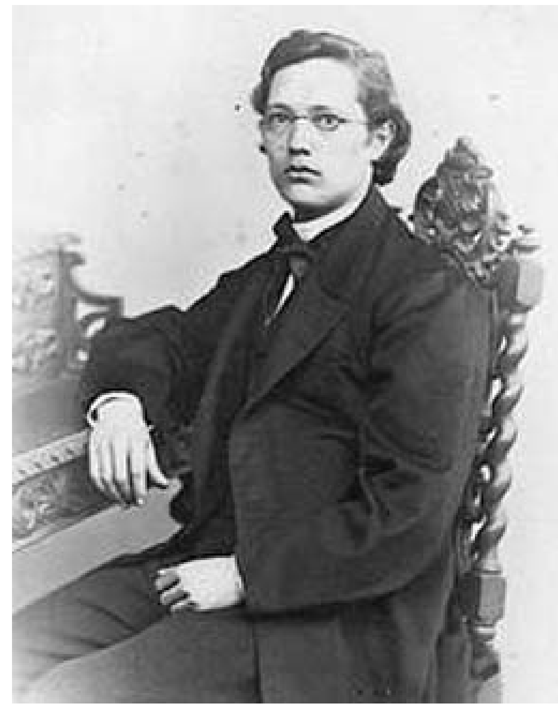
- K_C, the canonical divisor class on C is the class of $\operatorname{div}(\omega) \in \operatorname{Pic}(C)$ for any non-zero differential $\omega \in \Omega_C$.
  - $\to$ on $\mathbb{P}_1$, $K_{\mathbb{P}_1} = [-2 \cdot ([1,0])]$.
  - $\to$ on E, $K_E = [0]$.

... TODAY ...

RIEMANN - ROCH!

TODAY... RIEMANN-ROCH! (AND HURWITZ!)

Bernhard Riemann
(1826 – 1866)
age 39



Gustav Roch
(1839 – 1866)
age 26 (!!)

Chemistry
Electromagnetism (Ms)
"Abelian functions" (PhD)
Generalized Riemann's thm., and Max Noether refers to it as the Riemann-Roch theorem.
Died of "Consumption" aka tuberculosis
age 26.

RIEMANN - ROCH

DEF.
- $D = \sum_{P \in C} n_P \cdot (P)$ is positive (effective) $\iff n_P \geq 0 \ \forall P \in C$ (or $D \geq 0$)
- $D_1 \geq D_2 \iff D_1 - D_2 \geq 0$.
- $D \in \mathrm{Div}(C)$, $\mathcal{L}(D) = \{ f \in \bar{K}(C)^* : \mathrm{div}(f) \geq -D \} \cup \{0\}$

$$\ell(D) = \dim_{\bar{K}} \mathcal{L}(D).$$

ex $D = (\mathcal{O})$ $\quad \mathrm{div}(f) \geq -D \implies \mathrm{div}(f) \geq -(\mathcal{O})$

$$\implies \mathrm{div}(f) + (\mathcal{O}) \geq 0 \implies \underbrace{\sum_{P \in C} n_P \cdot (P) + (\mathcal{O})}_{=\ \sum_{P \neq \mathcal{O}} n_P \cdot (P) + (n_{\mathcal{O}}+1)\cdot(\mathcal{O})} \geq 0$$

$$\sum_{P \neq \mathcal{O}} n_P \cdot (P) + (n_{\mathcal{O}}+1)\cdot(\mathcal{O}) \geq 0$$

$\implies$ $n_P \geq 0 \quad \forall P \neq \mathcal{O}$ $\rightsquigarrow$ no poles at $P \neq \mathcal{O}$

$n_{\mathcal{O}} \geq -1 \quad P = \mathcal{O}$ $\rightsquigarrow$ a pole of order AT MOST 1 at $\mathcal{O}$.

PROP. (a) If $\deg D < 0 \Rightarrow \mathcal{L}(D) = \{0\}$, $\ell(D) = 0$.

$$\left(\mathcal{L}(D) = \{f \in \overline{K}(C)^* : \operatorname{div}(f) \geq -D\} \cup \{0\}\right)$$

(b) $\mathcal{L}(D)$ is a fin. dim'l $\overline{K}$-v.s. $\left(\ell(D) := \dim_{\overline{K}} \mathcal{L}(D)\right)$

(c) If $D \sim D'$ (i.e. $D - D' = \operatorname{div}(g)$) then $\mathcal{L}(D) \simeq \mathcal{L}(D')$, $\ell(D) = \ell(D')$.

<u>ex</u> What is $\mathcal{L}(0)$, $\ell(0)$? $\operatorname{div}(f) \geq -0 = 0 \Rightarrow \operatorname{div}(f)$ is effective $\Rightarrow n_P \geq 0$.

$\operatorname{div}(f) = \sum n_P (P)$

$\mathcal{L}(0) = \overline{K}$, $\ell(0) = 1$.

THEOREM. (Riemann-Roch) C smooth curve, $K_C = \operatorname{div}(\omega)$, $\omega \neq 0$, a canonical divisor.

There is an integer $g \geq 0$ called the genus of C such that for all $D \in \operatorname{Div}(C)$:

$$\ell(D) - \ell(K_C - D) = \deg D - g + 1.$$

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Cor. (a) $\ell(K_C) = g$ $\quad \left(D = 0. \quad 1 - \ell(K_C) = 0 - g + 1 \right)$

(b) $\deg(K_C) = 2g - 2$ $\quad \left(D = K_C. \quad g - 1 = \deg K_C - g + 1 \right)$

(c) If $\deg D > 2g-2 \Rightarrow \ell(D) = \deg D - g + 1.$

ex $\mathbb{P}^1$, $\overline{K}(\mathbb{P}^1) = \overline{K}(x)$, $\Omega_{\mathbb{P}^1} = \langle dx \rangle$, $K_{\mathbb{P}^1} = \underbrace{-2\cdot(\overset{([1,0])}{\overset{\shortparallel}{\infty}})} = \operatorname{div}(dx)$

$\deg(K_{\mathbb{P}^1}) = 2g(\mathbb{P}^1) - 2 = -2 \Longrightarrow \boxed{g(\mathbb{P}^1) = 0.}$

ex $C: y^2 = (x-e_1)(x-e_2)(x-e_3)$ smooth $(e_i \neq e_j)$

$\frac{dx}{y}$ is non-zero on Ω_C, $\operatorname{div}\left(\frac{dx}{y}\right) = 0 \Rightarrow K_C = 0 \Rightarrow \overset{g}{\overset{\shortparallel}{\ell(K_C)}} = \ell(0) = 1$

$\Rightarrow \boxed{g = 1}$

Also by RR, if $\deg D > 0 = 2\cdot 1 - 2 \Rightarrow \boxed{\ell(D) = \deg D}$

ex $C: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ $a_i \in K$ smooth.

you can show $\omega = \dfrac{dx}{2y + a_1x + a_3} \in \Omega_C$ is also non-vanish holom. and $\deg(\mathrm{div}(\omega)) = 0$.

$\overset{RR}{\Longrightarrow} g(C) = 1$.

PROP. (III. 3.1) $\mathcal{O} = [0,1,0]$

✓ (a) If $\underbrace{C: y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6}_{\text{Weierstrass model}}$ is a smooth curve $/K$, then $g = 1$.

(b) Let E be a smooth curve of genus 1 $/K$, w/ a point $\mathcal{O}$ defined over K.
Then there exist functions $x, y \in K(E)$ s.t.

$$\phi : E \longrightarrow \mathbb{P}^2$$
$$P \longmapsto [x(P), y(P), 1]$$

is an isomorphism of E/K onto a curve given by $C: y^2 + a_1xy + a_3y = \dots$ as in (a).
s.t. $\phi(\mathcal{O}) = [0,1,0]$.

(c) Any two Weierstrass models for E as in (a), (b) are related by a linear change of vars

$$x = u^2x' + r$$
$$y = u^3y' + su^2x' + t$$

w/ $u, r, s, t \in K$, $u \neq 0$.

PROOF (b).

Let E be a smooth curve of genus 1 over k, w/ $\mathcal{O} \in E(k)$.

Consider $\mathcal{L}(n\cdot(\mathcal{O}))$ for $n \geq 1$. (Recall: genus 1 $\overset{RR}{\Longrightarrow}$ if $\deg D > 0$ then $\ell(D) = \deg D$)

- $\ell((\mathcal{O})) = 1 \Rightarrow \mathcal{L}((\mathcal{O})) = \overline{K}$
- $\ell(2\cdot(\mathcal{O})) = 2 \Rightarrow \mathcal{L}(2(\mathcal{O})) = \langle 1, x \rangle_{\overline{K}}$ (Prop. 5.8. $\Rightarrow$ $x \in K(E)$)

NOTE: $x \in \mathcal{L}(2\mathcal{O}) \Rightarrow x$ has a pole of order at most 2 at $\mathcal{O}$

if $\mathrm{ord}_{\mathcal{O}} x = -1 \Rightarrow x \in \mathcal{L}((\mathcal{O})) \Rightarrow x \in \overline{K}$, $\mathcal{L}(2\mathcal{O}) = \langle 1 \rangle$!! $\times$

$\Rightarrow \mathrm{ord}_{\mathcal{O}} x = -2$.

- $\ell(3\cdot(\mathcal{O})) = 3 \Rightarrow \mathcal{L}(3\cdot(\mathcal{O})) = \langle 1, x, y \rangle$ SAME: $\mathrm{ord}_{\mathcal{O}}(y) = -3$.
- $\ell(4(\mathcal{O})) = 4 \Rightarrow \mathcal{L}(4\cdot(\mathcal{O})) = \langle 1, x, y, x^2 \rangle$
- $\ell(5(\mathcal{O})) = 5 \Rightarrow \mathcal{L}(5\cdot(\mathcal{O})) = \langle 1, x, y, x^2, xy \rangle$
- $\ell(6\cdot(\mathcal{O})) = 6 \Rightarrow \mathcal{L}(6\cdot(\mathcal{O})) = \langle \underbrace{1, x, y, x^2, xy, x^3, y^2}_{7 \text{ fns} \Rightarrow \text{ linear rel'n!!}} \rangle$

dim 6

$\mathcal{L}(6\cdot(O)) = \langle 1, x, y, x^2, xy, x^3, y^2 \rangle$

$\overset{l.d/k}{\Longrightarrow}$ $A_1 + A_2 x + A_3 y + A_4 x^2 + A_5 xy + \underbrace{A_6 x^3}_{\text{pole order } 6} + \underbrace{A_7 y^2}_{\text{pole order } 6} = 0.$

for some $A_i \in K$.

$\Rightarrow A_6 \cdot A_7 \neq 0$ change : $\begin{cases} x \mapsto -A_6 A_7 x \\ y \mapsto A_6 A_7^2 y \end{cases}$ and divide thru by $A_6^3 A_7^4$

$\rightsquigarrow$ a Weierstrass eq'n. $C : y^2 + a_1 xy + \dots = x^3 \dots$

Then: $\phi : E \longrightarrow C \subseteq \mathbb{P}^2.$

$P \longmapsto [x(P), y(P), 1]$ $\Big\}$ iso ?? Prove $\deg \phi = 1.$

$(K(E) = K(x,y))$

- Consider $\psi_1 : E \longrightarrow \mathbb{P}^1$

$P \longmapsto [x(P), 1]$

$O \longmapsto [1, 0]$

x has a pole order 2 at O, no other poles ψ_1

$\sum_{P \in \psi_1^{-1}(Q)} e_\phi(P) = \deg \psi_1$ only $O \mapsto [1,0]$

$e_\phi(O) = 2$

$\Rightarrow \deg \psi_1 = 2.$

- $\psi_2 : E \to \mathbb{P}^1$

$P \mapsto [y(P), 1]$ $\Big\}$ similarly $\deg \psi_2 = 3$

- $\deg \psi_1 = 2 \Rightarrow [k(E) : k(x)] = 2$
- $\deg \psi_2 = 3 \Rightarrow [k(E) : k(y)] = 3$

$$\begin{tikzcd} & k(E) \arrow[d, dash, "d"', "\cup|\,?"] \arrow[ddl, bend right, dash, "2"'] \arrow[ddr, bend left, dash, "3"] & \\ & k(x,y) & \\ k(x) \arrow[ur, phantom, "\subseteq"] & & k(y) \arrow[ul, phantom, "\supseteq"] \end{tikzcd}$$

$d \mid 2,\ d \mid 3 \Rightarrow d = 1$

$\Rightarrow [k(E) : k(x,y)] = 1.$

$\Rightarrow \deg \phi = 1$, ϕ is an isomorphism $E \cong \underbrace{C}_{\text{Weier. eq'n.}}$

(c) Suppose $C, C'/k$ are curves in Weier. model $C \cong_k C'$

$\{x, y\}$ $\{x', y'\}$

$\Rightarrow \{1, x\}, \{1, x'\}$ are bases for $\mathcal{L}(2 \cdot (O))/k$

$\{1, x, y\}, \{1, x', y'\}$ are bases for $\mathcal{L}(3 \cdot (O))/k$

$\Rightarrow x = u_1 x' + r$ $\qquad u_1, r, s_2, u_2, t \in k$

$y = u_2 y' + s_2 x' + t$ $\qquad$ and $y^2 \ldots = x^3 \ldots \Rightarrow u_1^3 = u_2^2$

let $u = u_2/u_1$, $s = s_2/u^2$ ✓

THEOREM. (Hurwitz)

Let $\phi : C_1 \to C_2$ be a non-ct. separable map of smooth curves.

genus g_1 genus g_2

Then

$$2g_1 - 2 \geq (\deg \phi) \cdot (2g_2 - 2) + \sum_{P \in C_1} (e_\phi(P) - 1).$$

Moreover, equality:

(i) $\operatorname{char}(k) = 0$

(ii) $\operatorname{char}(k) = p > 0$ and $p \nmid e_\phi(P)$ $\forall P \in C_1$.

ex

$$\phi : \overset{E:}{\{ y^2 = (x-e_1)(x-e_2)(x-e_3) \}} \longrightarrow \mathbb{P}^1 \quad /\mathbb{Q}$$

$$[x, y, z] \longmapsto [x, z]$$

Ramified at 4 pts: $P_1, P_2, P_3, \mathcal{O}$, $e = 2$.

$$2g_1 - 2 = 2 \cdot (2 \cdot 0 - 2) + (2-1) + (2-1) + (2-1) + (2-1)$$

$$= -4 + 4 = 0 \implies \boxed{g_1 = 1}$$