

EVEN MORE ON FORMAL GROUPS...

PREVIOUSLY...

- Formal groups (\mathcal{F}, F) R complete local ring, $\mathcal{M} \in R$:
Groups assoc. to formal gps $\mathcal{F}(\mathcal{M})$.
 $x \oplus y = F(x, y)$.
- PROP. Let $p = \text{char}(k) = \text{char}(R/\mathcal{M})$.
Then, every torsion elt. of $\mathcal{F}(\mathcal{M})$ has order a power of p .
- The invariant differential $\omega(T) = P(T)dT$ s.t. $\omega(F(T, S)) = \omega(T)$.
 \hookrightarrow Cor. $[p](T) = p \cdot f(T) + g(T^p)$ where $f, g \in R[[T]]$, $f(0) = g(0) = 0$.
- Formal log: $\log_{\mathcal{F}}(T) = \int \omega(T)$
 \hookrightarrow Unique inverse: $\exp_{\mathcal{F}}(T)$

Prop \mathcal{F}/R , $\text{char}(R) = 0$.

Then, $\log_{\mathcal{F}} : \mathcal{F} \rightarrow \widehat{G}_a$ is an isom. of formal gps over $K = R \otimes \mathbb{Q}$.

Pf. Let $\omega_{\mathcal{F}}(T)$ the formal inv. diff., so $\int \omega(F(T, S)) = \int \omega(T)$

$$\Rightarrow \log_{\mathcal{F}} F(T, S) = \log_{\mathcal{F}}(T) + C(S), \quad (*)$$

for some constant of integration $C(S) \in K[[S]]$.

Evaluate $T=0$, get $\log_{\mathcal{F}}(F(0, S)) = \log_{\mathcal{F}}(0) + C(S)$

$$\log_{\mathcal{F}}(S) \quad \Rightarrow \quad C(S) = \log_{\mathcal{F}}(S)$$

$(*) \Rightarrow \log_{\mathcal{F}}(F(T, S)) = \log_{\mathcal{F}}(T) + \log_{\mathcal{F}}(S)$, so it's a hom.

and $\exp_{\mathcal{F}}$ is inverse! $\Rightarrow \log_{\mathcal{F}}$ is an iso. \square

Prop (5.5) Let R be of $\text{char}(R)=0$. \mathcal{F}/R formal gp.

$$\text{Then, } \log_{\mathcal{F}}(T) = \sum_{n=1}^{\infty} \frac{a_n}{n} T^n, \quad \exp_{\mathcal{F}}(T) = \sum_{n=1}^{\infty} \frac{b_n}{n!} T^n$$

for some $a_n, b_n \in R$, $a_1 = b_1 = 1$.

F. GPs OVER DVR'S

Def (DVR) A discrete valuation ring is a ring R w/ ^{w/ fact. field K}
a valuation $\nu: K \rightarrow \mathbb{Z} \cup \{\infty\}$

$$\text{s.t. } \bullet \nu(x) = \infty \iff x=0$$

$$\bullet \nu(xy) = \nu(x) + \nu(y)$$

$$\bullet \nu(x+y) \geq \min\{\nu(x), \nu(y)\} \quad (\text{w/ eq. if } \nu(x) \neq \nu(y))$$

R complete^{local} wrt \mathcal{M} , max'l ideal, $x \in R$
 $\nu(x) = \text{largest } d \in \mathbb{Z} \text{ s.t. } x \in \mathcal{M}^d$

$\mathcal{F}/R \leftarrow$ a DVR, $\mathcal{F}(m)$ has no torsion of order prime to p .

GOAL Study the p -primary torsion of $\mathcal{F}(m)$. $\text{char}(R/m)$

Lemma R , $\text{char}(R)=0$, complete wrt ν (val), p prime, $\nu(p) > 0$.

(a) $f(T) = \sum_{n=1}^{\infty} \frac{a_n}{n} T^n$, $a_n \in R$ and $\nu(x) > 0$, then $f(x)$ converges in R .

(b) $g(T) = \sum_{n=1}^{\infty} \frac{b_n}{n!} T^n$, $b_n \in R$, $b_1 \in R^*$

if $\nu(x) > \frac{\nu(p)}{p-1}$ then $g(x)$ converges in R and $\nu(g(x)) = \nu(x)$.

ex $R = \mathbb{Q}_p$, $m = p\mathbb{Z}_p$, $\nu(p) = 1$, ex 1

