

# ELLIPTIC CURVES OVER LOCAL FIELDS

$K$  local field, complete wrt. a discrete valuation  $\nu$

$R$  ring of integers of  $K = \{x \in K : \nu(x) \geq 0\}$

$R^\times$  unit gp of  $R = \{x \in K : \nu(x) = 0\}$

$\mathcal{M}$  max'l ideal of  $R = \{x \in K : \nu(x) > 0\}$

$\pi$  a uniformizer for  $R$ ,  $\mathcal{M} = \pi R$ .

$k = R/\mathcal{M}$  the residue field,  $p = \text{char}(k)$

ex  $K = \mathbb{Q}_p$ ,  $\nu = \nu_p$  st.  $\nu_p(a) = \nu_p(m \cdot p^n) = n$

$R = \mathbb{Z}_p$ ,  $R^\times = \mathbb{Z}_p - p\mathbb{Z}_p = \{\alpha \in \mathbb{Z}_p : \alpha \not\equiv 0 \pmod{p}\}$

$\mathcal{M} = p\mathbb{Z}_p$ ,  $\pi = p$

$k = \mathbb{Z}_p / p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$

$a \in \mathbb{Z}$

$a = mp^n$ ,  $\text{gcd}(m, p) = 1$



$$E/K : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \quad a_i \in K$$

the change of variables:

preserves Weier. forms

$$\begin{cases} x = u^2 x' + r \\ y = u^3 y' + u^2 s x' + t \end{cases}$$

and

$$\Delta_{E'} = u^{12} \Delta_E$$

$$C_{4,E'} = u^4 \cdot C_{4,E}$$

In particular:

$$\left\{ \begin{array}{l} E : y^2 + a_1 xy + a_3 y = x^3 + \dots \iff E' : y'^2 + a'_1 x' y' + a'_3 y' = x'^3 + \dots \\ (x, y) \iff (u^{-2} x, u^{-3} y) \\ a_i \iff a'_i = u^i a_i \\ \Delta_E \iff \Delta_{E'} = u^{-12} \Delta_E \\ C_{4,E} \iff C_{4,E'} = u^{-4} C_{4,E} \end{array} \right.$$

a local field!!

Def A Weier. eq'n over  $K$  is called minimal (at  $\nu$ ) if

- $a_i \in R$
- $\nu(\Delta)$  is smallest among Weier. models w/  $a_i \in R$ .

ex  $E: y^2 + xy + y = x^3 + x^2 + 22x - 9 / \mathbb{Q} \subseteq \mathbb{Q}_p$

$$\Delta_E = -2^{15} \cdot 5^2, \quad C_{4,E} = -5 \cdot 211$$

$\Rightarrow E/\mathbb{Q}_p$  is minimal (for any  $p \in \mathbb{Z}$ )

Remark Although minimality is defined wrt  $v(\Delta)$ , the valuation of  $C_4$  helps decide whether  $v(\Delta)$  is as small as it can be subject to  $a_i \in \mathbb{R}$ .

ex  $E: y^2 + 6xy + 864y = x^3 + 180x^2 + 14256x + 559872$

$$\Delta_E = -2^{12} \cdot 3^{12} \cdot 11 \cdot 941 \quad (\text{Magma: Minimal Model}(E);)$$

$$[u, r, s, t] = [6, -72, 0, -216]$$

$$\rightarrow E': y^2 + xy + 4y = x^3 - x^2 + 2x - 1 \quad \Delta' = -11 \cdot 941.$$

$$\cong E'': y^2 + xy + 3y = x^3 + 2x^2 + 4x + 5 \quad \Delta'' = \Delta' = -11 \cdot 941.$$

Prop. (a) Every

















