

ELLIPTIC CURVES OVER LOCAL FIELDS

• THE ACTION OF INERTIA

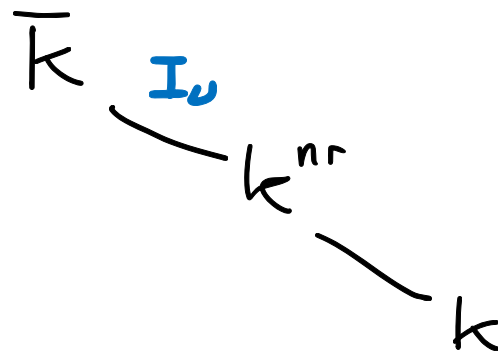
K local field, complete wrt v .

\bar{K} algebraic closure

K^{nr} maximal unramified ext'n of K in \bar{K}

$$G_K = \text{Gal}(\bar{K}/K) = D(\mathcal{M}_{\bar{K}}/\mathcal{M}_K)$$

$$I_v = \text{Gal}(\bar{K}/K^{nr}) = I(\mathcal{M}_{\bar{K}}/\mathcal{M}_K)$$



decomposition group

the inertia subgp.

$\sigma \in G_K$ s.t.
 $\sigma(\alpha) \equiv \alpha \pmod{\mathcal{M}_{\bar{K}}}$
 for all $\alpha \in R_{\bar{K}}$
 $\Rightarrow \tilde{\sigma} = \text{id on } \bar{k}$

$$1 \longrightarrow \text{Gal}(\bar{K}/K^{nr}) \longrightarrow \text{Gal}(\bar{K}/K) \longrightarrow \text{Gal}(K^{nr}/K) \longrightarrow 1$$

$$\quad \quad \quad \parallel \quad \quad \quad \parallel$$

$$\quad \quad \quad I_v \quad \quad \quad \text{Gal}(\bar{k}/k)$$

Def. Σ is unramified at v if I_v acts trivially on Σ .

Prop. Let E/k be an ell. curve, s.t. \tilde{E}/k is non-singular. (GOOD REDUCTION!)

(a) Let $m \geq 1$ s.t. $\gcd(m, \text{char}(k)) = 1$.

Then $E[m]$ is unramified at v .

(b) Let $l \neq \text{char}(k)$, then $T_l(E)$ is unramified at v .

Q: Converse? THE CRITERION OF NÉRON-OGG-SHAFAREVICH.

Thm. E/K ell. curve. TFAE:

(a) E has good reduction over K

(b) $E[m]$ is unramified at v for all integers $m \geq 1$ rel. prime to $\text{char}(k) = p$

(c) The Tate module $T_l(E)$ is unramif. at v for some (all) primes l rel. prime to p .

(d) $E[m]$ is unramif. at v for only many integers $m \geq 1$ rel. pr. to p .

GOOD AND BAD REDUCTION

$$E/k \xrightarrow{\sim} \tilde{E}/k$$

min'l model

Def.

(a) E has good (or stable) reduction over k if \tilde{E} is non-singular.

(b) E has **multiplicative** (or semi-stable) reduction over k if

\tilde{E} has a node  two "tangent" lines at sing.

(b.1) If the slopes are in k then, it is split mult.

(b.2) If the slopes are not in k , it is non-split mult.

(c) E has **additive** red'n (or unstable) if \tilde{E} has a cusp  (only one "tangent" line)

Prop. (III.1.4, III.2.5)

(HOMEWORK!)

Let E/K be given by a minimal Weier. eq'n $y^2 + a_1xy + \dots = x^3 + \dots$

Let Δ_E be the disc., and C_4, c be as usual.

(a) E has good red'n $\iff v(\Delta) = 0$ (i.e., $\Delta \in R^\times$)

In this case \tilde{E}/\tilde{k} is an ell. curve.

(b) E has mult. red'n $\iff v(\Delta) > 0$, $v(C_4) = 0$.

In this case $\tilde{E}_{ns}(\bar{k}) \cong \bar{k}^*$

(c) E has add. red'n —

