A NOTE ABOUT POTENTIAL GOOD REDUCTION

Recall: E/K has potential good reduction at v if there is a finite extension K'/K s.t. $\tilde{E}/_{k'}$ is smooth.

Prop. Let E/K be an elliptic curve. Then:
E has potential good reduction if and only if $j(E)$ is integral ($j(E) \in R$).

(E has mult red'n if and only if $j(E)$ is non-integral at v
i.e., $v(j(E)) < 0$)

Proof.
(char $k \neq 2$) We may assume that E/k is given by a Legendre normal form:

$$E: \ y^2 = x(x-1)(x-\lambda), \quad \lambda \neq 0, 1.$$

($\Leftarrow$) Assume $\boxed{j(E) \in R}$

$$j(E) = \frac{(1-\lambda(1-\lambda))^3}{\lambda^2(1-\lambda)^2} \Rightarrow (1-\lambda(1-\lambda))^3 - j\lambda^2(1-\lambda)^2 = 0 \text{ and } j \in R$$

$\rightarrow$ coeffs are in R, monic in λ

$\Rightarrow \lambda \in R$, and $\lambda \not\equiv 0, 1 \bmod m \Rightarrow \Delta = 16\lambda^2(\lambda-1)^2 \not\equiv 0 \bmod m$

$\Downarrow$ $1 \equiv 0 \bmod m$ ※ $\quad \Rightarrow$ model has good red'n! ▨

($\Rightarrow$) <u>Assume pot. good red'n!</u>

Let K'/K finite ext'n s.t. E/K' has good red'n.

$$\Rightarrow \begin{cases} \Delta' \in R'^{\times} \\ c_4' \in R' \end{cases} \text{ and } j(E) = \frac{(c_4')^3}{\Delta'} \in R'$$

But $j(E) \in K \Rightarrow j(E) \in K \cap R' = R \Rightarrow j(E) \in R$ ▨

Remark FACT: $K/\mathbb{Q}_p$ finite, and if E/K has CM then E/K has pot. good red'n. $\Rightarrow j(E) \in R$.

Thus: if $E/\mathbb{Q}$ has CM then $j(E) \in \mathbb{Z}_p$ for all p $\Rightarrow j(E) \in \mathbb{Z}$.

ex $y^2 = x^3 - x$ / $\mathbb{Q}$ has CM (by $\mathbb{Z}[i]$) and $j(E) = 1728$ | $y^2 = x^3 + 1$ / $\mathbb{Q}$ has CM (by $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$) and $j(E) = 0$.