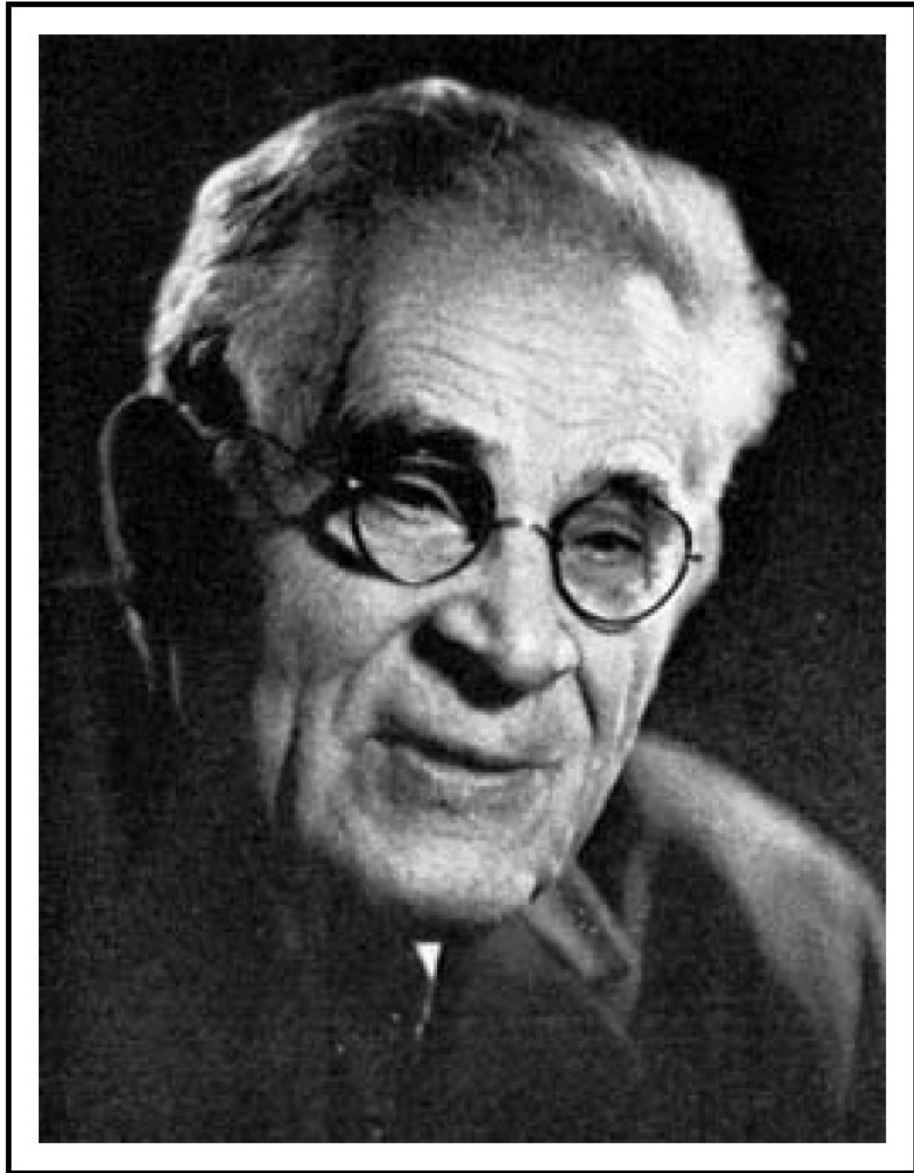


THE

MORDELL

- WEIL

THEOREM



Louis Mordell  
1888 – 1972 /  $\oplus$   
(1922)



André Weil  
1906 – 1998 / ab vars  
over # flds  
(1928)

Thm Let  $E/K$  be an elliptic curve,  
where  $K$  is a number field.

Then,  $E(K)$  is a finitely generated abelian group.

i.e.,  $E(K) = \langle P_1, \dots, P_n \rangle_{\mathbb{Z}}$  for some  $P_i \in E(K)$

In particular,  $E(K) \cong E(K)_{\text{tors}} \oplus \mathbb{Z}^{R_{E/K}}$

where  $E(K)_{\text{tors}}$  is a finite subgroup formed by points of finite order  
and  $R_{E/K}$  is a non-neg. integer, called the rank of  $E/K$ .

( $R_{E/K} = \text{rank}_{\mathbb{Z}} E(K)$ .)

# Strategy.

- Weak Mordell-Weil thm:

(NOTE: If  $E(k) \cong T \oplus \mathbb{Z}^R \Rightarrow m > 2$   $E(k)/_m E(k) \cong \underbrace{T/mT \oplus (\mathbb{Z}/m\mathbb{Z})^R}_{\text{is finite}}$ )

$k$  a number field,  $m > 2$ , then  $E(k)/_m E(k)$  is a finite gp.

- Descent procedure:

Let  $A$  be an abelian gp w/ a "height" function, and there is  $m > 2$  st.  $A/_m A$  is finite. Then,  $A$  is finitely generated.

## Notation

$K$  a number field

$M_K$  a complete set of inequivalent absolute values on  $K$ .

$\hookrightarrow M_K^\infty$  the archimedean abs. values on  $K$

$$([K:\mathbb{Q}] = r_1 + 2r_2, \quad K \xrightarrow{r_1} \mathbb{R}, \quad K \xrightarrow{2r_2} \mathbb{C})$$

$\hookrightarrow M_K^o$  the non-archimedean abs values on  $K$

$$(\mathfrak{p} \subseteq \mathcal{O}_K \text{ then } |x|_{\mathfrak{p}} = \frac{1}{p^{v_{\mathfrak{p}}(x)}})$$

$v(x) = -\log |x|_v$  for abs. values  $v \in M_K$

$ord_v$  normalized val. for  $v \in M_K^o$  (i.e.  $ord_v(K^*) = \mathbb{Z}$ )

$R$  or  $\mathcal{O}_K$  ring of integers of  $K = \{x \in K : v(x) \geq 0 \ \forall v \in M_K^{\circ}\}$

$R^{\times}$  or  $\mathcal{O}_K^{\times}$  unit gp of  $R = \{x \in K : v(x) = 0 \ \forall v \in M_K^{\circ}\}$

$K_{\nu}$  completion of  $K$  at  $\nu$ , for  $\nu \in M_K$ .

$R_{\nu}, \mathcal{M}_{\nu}, k_{\nu}$ .

Thm (Weak Mordell-Weier)













