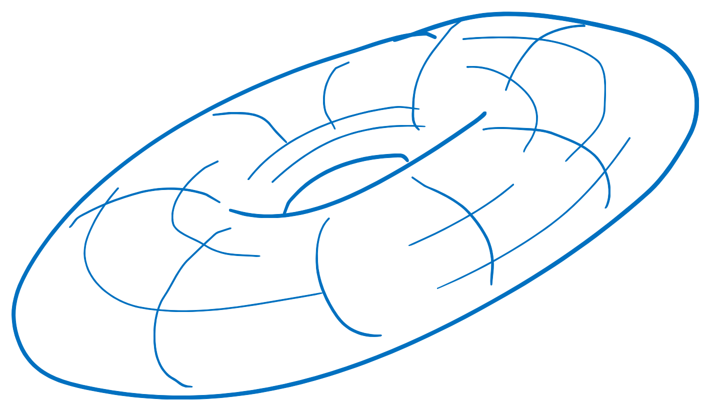
π and elliptic curves



$$e^{\pi\cdot\sqrt{163}} = 262537412640768743.9999999999992500725\ldots$$

12 nines!

transcendental number!

Theorem (Gelfond-Schneider, 1934)

If a, b are algebraic numbers with $a \neq 0, 1$ and b irrational, then any value of a^b is a transcendental number.

ex $e^{\pi\sqrt{163}} = e^{i\pi \cdot i\sqrt{163}} = (-1)^{\sqrt{-163}}$ is transcendental!

$K = \mathbb{Q}(\sqrt{-163})$

$E_{163}: \quad y^2 + y = x^3 - 2174420x + 1234136692$

$\mathrm{End}(E_{163}) \cong \mathcal{O}_K \cong \mathbb{Z}\left[\frac{1+\sqrt{-163}}{2}\right]$ so E_{163} has CM.

($\Rightarrow$ E_{163} has potential good reduction at all primes)

$\Rightarrow j(E_{163}) \in \mathbb{Z}$

$$j(E_{163}) = -262537412640768000 = -2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$$

Also $E/\mathbb{C} \cong \mathbb{C}/\Lambda$ and $\Lambda = \langle \omega_1, \omega_2 \rangle$ is unique up to homothety!

$$\mathbb{C}/\Lambda \cong \mathbb{C}/\langle 1, \frac{\omega_1}{\omega_2} \rangle \cong \mathbb{C}/\langle 1, \frac{\omega_2}{\omega_1} \rangle$$

$\tau \in \mathbb{H}$

$\cong \mathbb{C}/\langle 1, \tau \rangle \quad \tau \in \mathbb{H}.$

Turns out: $E_{163}/\mathbb{C} \Rightarrow \Lambda = \left\langle 1, \frac{1+\sqrt{-163}}{2} \right\rangle$

generate$/_{\mathbb{Z}}$ $\mathcal{O}_K$, w/ $K = \mathbb{Q}(\sqrt{-163})$.

One can give an expression:

$$j(E) = j(\tau) = j(q) \qquad q = e^{2\pi i \tau}$$

$E/\mathbb{C} \cong \mathbb{C}/\langle 1, \tau \rangle$

$$j(q) = \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + \cdots$$

(q-expansion)

In our case $\tau = \dfrac{1+\sqrt{-163}}{2}$

$$q = e^{\pi i (1+\sqrt{-163})} = e^{\pi i} \cdot e^{\pi i \sqrt{-163}}$$

$$= - e^{-\pi\sqrt{163}} = - \frac{1}{e^{\pi\sqrt{163}}}$$

$$\cong -3.809 \cdot 10^{-18}$$ (VERY SMALL!)

Thus!

$$j(E_{163}) \approx \frac{1}{q} + 744 \quad \Big(\underbrace{+\, 196884 \cdot q + \dots}_{\text{error} \approx 10^{-12}}\Big)$$

$$\Rightarrow \underbrace{\underbrace{744}_{\in \mathbb{Z}} - \underbrace{j(E_{163})}_{\in \mathbb{Z}}}_{\in \mathbb{Z}} \cong -\frac{1}{q} = e^{\pi\sqrt{163}}$$

$e^{\pi\sqrt{163}}$ is an integer (up to an error of 10^{-12})