

# The Weak! Mordell-Weil Theorem (part 2)

PROP Let  $K$  be a number field,  $S \subseteq M_K$  a finite set of places containing  $M_K^\infty$ , and let  $m \geq 2$  be an integer. Let  $L/K$  ( $L = K_{S,m}^{ab}$ ) be the maximal abelian extension of  $K$  having exponent  $m$  which is unramified outside of  $S$ . Then,  $L/K$  is a finite extension.

Thm (Weak Mordell-Weil)

Let  $K$  be a number field,  $m \geq 2$ ,  $E/K$  an elliptic curve. Then,  $E(K)/{}_mE(K)$  is finite.

Proof. The existence of the Kummer pairing

$\Rightarrow (E(K)/{}_mE(K))$  is finite  $\iff \text{Gal}(L/K)$  is finite w/  $L = K(\sqrt[m]{E(K)})$

We showed that  $L/K$  is <sup>abelian and exp  $m$</sup>  unramified outside  $S = M_K^\infty \cup \{v \text{ of bad red. for } E/K\} \cup \{v(m) \neq 0\}$

By prop  $\underbrace{K \subseteq L \subseteq K_{S,m}^{ab}}_{\text{finite}} \Rightarrow L/K$  is finite  $\Rightarrow E(K)/{}_mE(K)$  finite!  $\square$

# Remarks.

• Suppose  $E[m] \subseteq E(k) \cong E(k)_{\text{tors}} \oplus \mathbb{Z}^{R_{E/k}}$

$\Rightarrow E(k)/_m E(k) \cong \underbrace{\frac{E(k)_{\text{tors}}}{_m E(k)_{\text{tors}}}}_{\mathbb{Z}/m \oplus \mathbb{Z}/m} \oplus \left( \frac{\mathbb{Z}}{m\mathbb{Z}} \right)^{R_{E/k}}$

•  $\kappa : E(k)/_m E(k) \times \text{Gal}(L/k) \rightarrow E[m]$  perfect.

$\rightarrow E(k)/_m E(k) \hookrightarrow \text{Hom}(\text{Gal}(L/k), E[m])$   
 $\cong \text{Hom}(\text{Gal}(L/k), \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z})$   
 $\cong \text{Hom}(\text{Gal}(L/k), \mathbb{Z}/m\mathbb{Z})^2$

•  $\mu_m \subseteq K \subseteq L \subseteq K_{S,m}^{\text{ab}}$  where  $S = \{ \text{bad primes, } \infty \text{ primes, } \nu/m \}$

•  $K_{S,m}^{\text{ab}} = K(\sqrt[m]{a} : a \in T_S)$ , where  $T_S = \{ a \in K^{\times} / (K^{\times})^m : \text{ord}_{\nu}(a) \equiv 0 \pmod m \ \forall \nu \in M_k^{\circ} \setminus S \}$

Weird pairing!

- $$R_S^x \longrightarrow T_S$$

$$a \longmapsto a \bmod (K^x)^m$$

want this surjective ...

$$a \in T_S, a R_S = \mathfrak{L}^m$$

Need  $\mathfrak{L}$  to be principal

$$[\mathfrak{L}] \in \text{Cl}(K)[m] \cong \mathbb{Z}/m\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m\mathbb{Z}$$

$$\begin{array}{ccc} & \uparrow & \uparrow \\ & \mathfrak{f}_1 & \mathfrak{f}_r \end{array}$$

o add  $\mathfrak{f}_1, \dots, \mathfrak{f}_r$  to  $S$ , get  $S'$  (we have increased  $S$  by  $\text{rank}_{\mathbb{Z}/m\mathbb{Z}}(\text{Cl}(K)[m])$ )

$$o \mu_m \subseteq K \subseteq L \subseteq K_{S,m}^{ab} \subseteq K_{S',m}^{ab}$$

$$\text{and } K_{S',m}^{ab} = K(\sqrt[m]{a} : a \in C)$$



















