

The Néron-Tate Canonical Height

Def The (Néron-Tate) canonical height on $E(\mathbb{Q})$ is

$$\hat{h} : E(\mathbb{Q}) \rightarrow \mathbb{R}$$

defined by

$$\hat{h}(P) = \frac{1}{\deg f} \lim_{N \rightarrow \infty} \frac{h_f([2^N]P)}{4^N}$$

where f is any non-constant even function on E (e.g., the x -coordinate).

Note: We proved the limit exists and it is independent of the choice of f .

Thm. E/\mathbb{Q} , with can. height \hat{h} .

(e) $f \in \mathbb{Q}(E)$ even, then $(\deg f) \cdot \hat{h} = h_f + O(1)$ constant depends only on E, f .

(a) $\forall P, Q \in E(\mathbb{Q})$

$$\hat{h}(P+Q) + \hat{h}(P-Q) = 2\hat{h}(P) + 2\hat{h}(Q) \quad (\text{"parallelogram law"})$$

(b) $\forall P \in E(\mathbb{Q}), m \in \mathbb{Z}, \hat{h}([m]P) = m^2 \hat{h}(P)$.

(c) \hat{h} is a quad. form on E , i.e., \hat{h} is even and

$$\langle, \rangle : E(\mathbb{Q}) \times E(\mathbb{Q}) \rightarrow \mathbb{R}$$

$$(P, Q) \mapsto \langle P, Q \rangle = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q) \text{ is bilinear.}$$

(d) $P \in E(\mathbb{Q}), \hat{h}(P) \geq 0$ and $\hat{h}(P) = 0 \iff P$ is torsion. (!! useful !!)

If \hat{h}' satisfies (e) and (b) for any $m \geq 2$, then $\hat{h}' = \hat{h}$.

Pf $E/\mathbb{Q}, \hat{h}$.

(a) f even, then $(\deg f) \hat{h} = h_f + O(1)$

We proved: $\left| \frac{h_f([2^N]P)}{4^N} - \frac{h_f([2^M]P)}{4^M} \right| \leq \frac{C}{3 \cdot 4^M}$ for all $N \geq M \geq 0$.

Let $M=0, N \rightarrow \infty \Rightarrow |(\deg f) \hat{h}(P) - h_f(P)| \leq \frac{C}{3 \cdot 4} \quad \square$

(a) P-law. We showed:

$$\left(h_f(P+Q) + h_f(P-Q) = 2h_f(P) + 2h_f(Q) + O(1) \right) \times \frac{\deg f}{4^N}$$

$$\deg f \frac{h_f([2^N](P+Q))}{4^N} + \dots = \dots + 2 \deg f \frac{h_f([2^N]Q)}{4^N} + \frac{O(1)}{4^N} \rightarrow 0$$

$$N \rightarrow \infty \quad \hat{h}(P+Q) + \hat{h}(P-Q) = 2\hat{h}(P) + 2\hat{h}(Q). \quad \square$$

$$(b) \quad \underline{\hat{h}([m]P) = m^2 \hat{h}(P).}$$

$$\left(h_g([m]P) = m^2 h_g(P) + O(1) \right) \times \frac{\deg f}{4^N} \quad N \rightarrow \infty \quad \square$$

(c) • \hat{h} is even: put $P = \mathcal{O}$ in the par.-law $\overset{\mathcal{O}}{=}$

$$\hat{h}(Q) + \hat{h}(-Q) = 2\hat{h}(\mathcal{O}) + 2\hat{h}(Q) \Rightarrow \hat{h}(Q) = \hat{h}(-Q).$$

• $\langle P, Q \rangle := \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)$. is bilinear

Suffices to show $\langle P+R, Q \rangle \stackrel{?}{=} \langle P, Q \rangle + \langle R, Q \rangle$

$$\text{or } \hat{h}(P+R+Q) - \hat{h}(P+R) - \hat{h}(P+Q) - \hat{h}(R+Q) + \hat{h}(P) + \hat{h}(Q) + \hat{h}(R) \stackrel{?}{=} 0$$

Use the p-law: $\left. \begin{array}{l} + (P+R, Q) \\ - (P, R-Q) \\ + (P+Q, R) \\ - 2 \cdot (R, Q) \end{array} \right\} \Rightarrow \square$

$$(d) \quad h_g(P) = \log |H(f(P))| \geq 0 \Rightarrow \hat{h}(P) \geq 0.$$

$$\underbrace{P \text{ is torsion} \Rightarrow \hat{h}(P) = 0}_{\exists m \geq 1} \quad [m]P = O \Rightarrow \hat{h}(P) = \frac{\hat{h}([m]P)}{m^2} = \frac{\hat{h}(O)}{m^2} = 0. \quad \square$$

$$\underbrace{\hat{h}(P) = 0 \Rightarrow P \text{ is torsion}} \quad \hat{h}([m]P) = m^2 \hat{h}(P) = 0 \text{ for all } m \geq 1.$$

From (e) $\exists C \forall m \in \mathbb{Z}$

$$h_g([m]P) = |(\deg f) \hat{h}([m]P) - h_g([m]P)| \leq C.$$

$\Rightarrow \{P, [2]P, [3]P, \dots\} \subseteq \{Q \in E(\mathbb{Q}) : h_g(Q) \leq C\}$ is a finite set!

$$\Rightarrow [n]P = [m]P \Rightarrow [n-m]P$$

