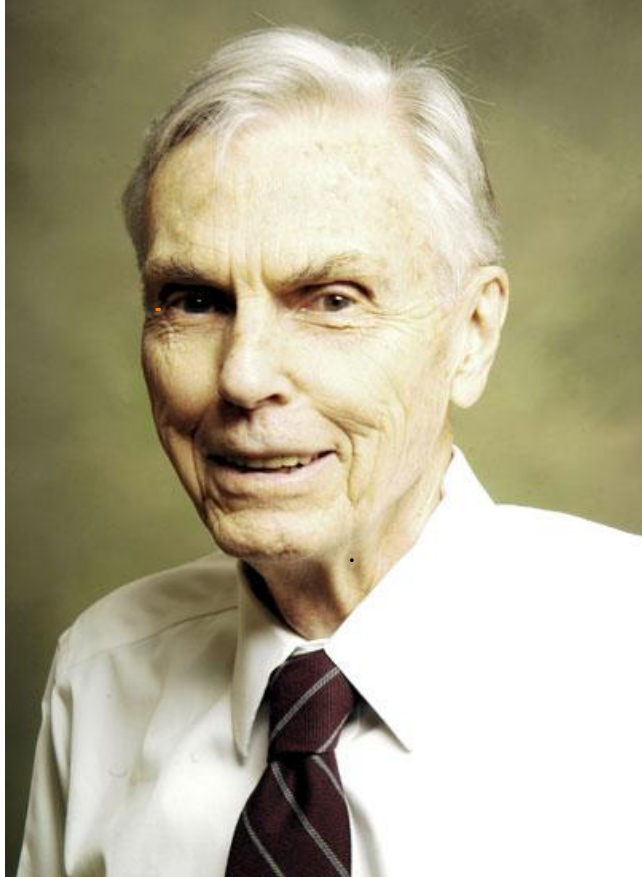


SELMER AND SHA: The fundamental sequence



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0 → MW → Selmer
(weak)
(1920 - 2006)
(Norway)

Tate
(1925 - 2019)
(USA)

Shafarevich → 0
(1923 - 2017)
(Russia)

E/k ell. curve, $\phi: E \rightarrow E'/k$ isogeny.

From $0 \rightarrow E[\phi] \rightarrow E \xrightarrow{\phi} E' \rightarrow 0$ using cohomology we get:

$$\begin{array}{ccccccc}
 0 & \rightarrow & \frac{E'(k)}{\phi(E(k))} & \xrightarrow{f} & H^1(G_k, E[\phi]) & \rightarrow & H^1(G_k, E)[\phi] \rightarrow 0 \\
 & & \downarrow & \curvearrowright & \downarrow & \curvearrowright & \downarrow \\
 0 & \rightarrow & \prod_{v \in M_k} \frac{E'(k_v)}{\phi(E(k_v))} & \xrightarrow{f_v} & \prod_{v \in M_k} H^1(G_v, E[\phi]) & \rightarrow & \prod_{v \in M_k} H^1(G_v, E)[\phi] \rightarrow 0
 \end{array}$$

where $G_v = \text{Gal}(\bar{k}_v/k_v)$.

Def'n $S^{(\phi)}(E/k) = \text{Ker} \left\{ H^1(G_k, E[\phi]) \rightarrow \prod_{v \in M_k} H^1(G_v, E) \right\}$ ϕ -Selmer gp.

III $(E/k) = \text{Ker} \left\{ H^1(G_k, E) \rightarrow \prod_{v \in M_k} H^1(G_v, E) \right\}$ Tate-Shafarevich gp.

III $(E/k)[\phi]$

Thm (a) There is an exact sequence

$$0 \rightarrow \frac{E'(k)}{\phi(E(k))} \rightarrow S^{(\phi)}(E/k) \rightarrow \text{III}(E/k)[\phi] \rightarrow 0$$

(b) The Selmer gp. $S^{(\phi)}(E/k)$ is finite.

pf. (a) By def'n of $\text{Sha}[\phi]$.

(b) Let $\xi \in S^{(\phi)}(E/k)$, let $\nu \in M_k^o$, not dividing $m = \deg \phi$, s.t. E'/k has good red'n at ν ($\nu \notin "S"$). Then we claim that ξ is unramified at ν .

$\xi \in H^1(G_k, E[\phi])$ unramified means that $\xi: G_k \rightarrow E[\phi]$ and $\xi(\sigma) = 0 \forall \sigma \in I_\nu$, where $I_\nu \subseteq G_\nu$ is inertia at ν .

Since $\xi \in S^{(\phi)} \subseteq H^1(G_k, E[\phi]) \Rightarrow \xi$ is trivial in $H^1(G_\nu, E)$. (a coboundary!)

$$\Rightarrow \exists P \in E(\bar{k}_\nu) \text{ s.t. } \xi(\sigma) = \sigma(P) - P \quad \forall \sigma \in G_\nu$$

$$\xi \in S^{(g)} \Rightarrow \exists P \in E(\bar{k}_v) \text{ s.t. } \xi(\sigma) = \sigma(P) - P \quad \forall \sigma \in G_v.$$

$$\Rightarrow \xi_c$$

