

GROUP (AND GALOIS) COHOMOLOGY

Let G be a finite group, and let M be an abelian group on which G acts.

GOAL: $H^0(G, M)$ and $H^1(G, M)$

$M^{\mathcal{D}G}$, makes M into a G -module, $\sigma \in G$ we write $\sigma \cdot m = \sigma(m) = m^\sigma$

NOTE: G is a profinite group w/ a prof. topology and we require the action of G on M to be continuous wrt the prof. top. on G and the discrete top. on M .

examples

$G = \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{C}, \mathbb{C}^\times$

$\{1, c\}$ acts (trivially) on $\mathbb{R}, \mathbb{R}^\times$

ex L/k finite Galois ext'n of fields ^{number}

then $G = \text{Gal}(L/k)$ acts on $L, L^\times, \mathcal{O}_L, \mathcal{O}_L^\times$

ex $\bar{K}/k, \mu_m$ m -th roots of unity $\rightarrow \bar{K}^\times$, then $\text{Gal}(\bar{K}/k)$ acts on μ_m .

(note: $\bar{\mathbb{R}} = \mathbb{C}, \text{Gal}(\bar{\mathbb{R}}/\mathbb{R}) = \text{Gal}(\mathbb{C}/\mathbb{R})$.)

ex $\bar{K}/k, E/k$ ell. curve, $E[n] = \{P \in E(\bar{K}) : [n]P = \mathcal{O}\}$ $\curvearrowright \text{Gal}(\bar{K}/k)$

$\text{Gal}(\bar{K}/k)$ acts on $T_\mu(k) = \varprojlim \mu_n$ (Tate module)

on $T(E) = \varprojlim E[n]$

on $T_p(E) = \varprojlim E[p^n]$

- M is a G -module :
- $e \in G$ ident. , $e \cdot m = e(m) = m$.
 - $\sigma \in G$, $\sigma \cdot (m+m') = \sigma \cdot m + \sigma \cdot m'$
 - $(\sigma\tau) \cdot m = \sigma(\tau(m))$ NOTE! LEFT-MODULES!
 $\sigma, \tau \in G$

A G -module hom. b/w G -modules M, N is $\phi: M \rightarrow N$ is a hom.
 s.t. $\sigma \cdot (\phi(m)) = \phi(\sigma \cdot m)$
 for all $m \in M$, $\sigma \in G$.

Of interest:

$$M^G = \text{largest submodule of } M \text{ where } G \text{ acts trivially.}$$

$$= \{m \in M : \sigma \cdot m = m \quad \forall \sigma \in G\}$$

ex E/\mathbb{Q} , $G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $E[n]^G = E(\mathbb{Q})[n]$.

$\underbrace{\hspace{10em}}_{E(\mathbb{Q})[n]}$

ex $G = \text{Gal}(L/K)$, $L^G = K$, $M_n(L)^G = M_n(K)$.
 $O_L^G = O_K$

Now let P, M, N be G -modules and let

$$0 \rightarrow P \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$$

be an exact seq. of G -modules : $\begin{cases} \cdot \alpha, \beta \text{ are } G\text{-mod. hom.} \\ \cdot \alpha \text{ is injective, } \beta \text{ is surjective.} \\ \cdot \text{Im } \alpha = \text{Ker } \beta. \end{cases}$
 and apply G -invariance:

$$0 \rightarrow P^G \xrightarrow{\alpha'} M^G \xrightarrow{\beta'} N^G \quad \text{where } \alpha' = \alpha|_{P^G}, \beta' = \beta|_{M^G}$$

Q: Is this still exact?

- α' injective? α is injective, and $\alpha' = \alpha|_{P^G}$ is also injective.
- $\text{Im } \alpha' = \text{Ker } \beta'$?

$$0 \longrightarrow P^G \xrightarrow{\alpha'} M^G \xrightarrow{\beta'} N^G$$

- $\text{Im } \alpha' = \text{Ker } \beta' ?$

Note: • $\text{Ker } (\beta') = \text{Ker } (\beta|_{M^G}) = \text{Ker } \beta \cap M^G$

- $\text{Im } (\alpha') = \text{Im } (\alpha) \cap M^G ?$

$$\text{Im } (\alpha') \subseteq \text{Im } (\alpha) \cap M^G \quad \checkmark$$

$$\supseteq \quad m \in \text{Im } (\alpha) \cap M^G \Rightarrow \exists p \in P \text{ s.t. } \alpha(p) = m.$$

Now let $\sigma \in G$, $\sigma(\alpha(p)) = \sigma(m) = m$ ↙ $m \in M^G$

$$\begin{cases} \alpha(\sigma(p)) = m \\ \alpha(p) = m \end{cases} + \alpha \text{ inj} \Rightarrow p = \sigma(p) \Rightarrow p \in P^G.$$

\Rightarrow

