Math 5020 - Elliptic Curves
Homework 6

Problem 1. (Silverman's VIII: 8.1) Let K be a number field, E/K and elliptic curve, $m \geq 2$ and integer, $\mathrm{Cl}(K)$ the ideal class group of K and

$$S = \{\nu \in M_K^0 : E \text{ has bad reduction at } \nu\} \cup \{\nu \in M_K^0 : \nu(m) \neq 0\} \cup M_K^\infty.$$

Assuming that $E[m] \subset E(K)$, prove the following quantitative version of the weak Mordell-Weil theorem:

$$\mathrm{rank}_{\mathbb{Z}/m\mathbb{Z}}(E(K)/mE(K)) \leq 2\#S + 2\,\mathrm{rank}_{\mathbb{Z}/m\mathbb{Z}}\left(\mathrm{Cl}(K)[m]\right).$$

Note: Yes, *again*. This is may be the most important exercise in the whole course, so if you didn't prove it last time, you now have an extension to keep working on it. In order to solve it, you need to *really* understand the proof of the Weak Mordell-Weil theorem, so it is a very worthwhile exercise.

Problem 2. (Descent)

(a) Let $E/\mathbb{Q}$ be our old friend, the curve $y^2 = x^3 + 2x^2 - 3x$. First, "by hand", calculate $E(\mathbb{Q})_{\mathrm{tors}}$ (i.e. use VII.3.1 or VIII.7.2). Then use the method of complete 2-descent (Ch. X, §1) to show that the rank of E is 0.

(b) The elliptic curve $y^2 = x^3 - 82x$ has rank 3. First, "by hand", calculate $E(\mathbb{Q})_{\mathrm{tors}}$. Then use the method of descent via 2-isogeny (Ch. X, §4) to show that the rank is indeed 3, and to find generators for $E(\mathbb{Q})/2E(\mathbb{Q})$. In the process, describe explicitly what are the following objects in this case (as the appear in Prop. 4.9):

$$E,\ E',\ \phi,\ \hat{\phi},\ S,\ \mathbb{Q}(S,2),\ C_d,\ C_d',\ S^{\phi}(E/\mathbb{Q}),\ S^{\hat{\phi}}(E'/\mathbb{Q}),\ \psi,\ \psi'.$$

Are there any elements in $Ш(E/\mathbb{Q})[\phi]$ *or in* $Ш(E'/\mathbb{Q})[\hat{\phi}]$?

Problem 3. (Elliptic curves with non-trivial rank.) The goal here is a systematic way to find curves of rank at least $r \geq 0$, without using tables of elliptic curves:

(a) (Easy) Find 3 non-isomorphic elliptic curves over $\mathbb{Q}$ with rank ≥ 2. You must prove that the rank is at least 2. (To show linear independence, you may use PARI or SAGE to calculate the height matrix).

(b) (Fair) Find 3 non-isomorphic elliptic curves over $\mathbb{Q}$ with rank ≥ 3.

(c) (Medium difficulty) Find 3 non-isomorphic elliptic curves over $\mathbb{Q}$ with rank ≥ 6. If so, then you can probably find 3 curves of rank ≥ 8 as well.

(d) (Significantly harder) Find 3 non-isomorphic elliptic curves over $\mathbb{Q}$ of rank ≥ 10.