HYPERGEOMETRIC FUNCTIONS, CHARACTER SUMS AND APPLICATIONS

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Abstract. We summarize several aspects of hypergeometric functions based on our recent work [9, 8, 12, 16, 17, 18, 20, 21] and our understanding of the subjects.

1. Classical hypergeometric functions and differential equations

For a discussion on the topic, please see [1]. Our approach has overlaps with [12].

1.1. Gamma and beta functions.

Let $\mathbb{N} = \mathbb{Z}_{>0}$.

Recall the usual binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

For any $a \in \mathbb{C}$ and $n \in \mathbb{N}$, define the *rising factorial* or *Pochhammer symbol* by

$$(a)_n := a(a+1)\cdots(a+n-1), \tag{1}$$

and define $(a)_0 = 1$. Note then, that

$$\frac{(-1)^k(-n)_k}{k!} = \binom{n}{k}. \tag{2}$$

Functions *gamma function* $\Gamma(x)$ and *beta function* $B(x,y)$ are defined as follows.

Definition 1. For $\mathrm{Re}(x) > 0$,

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t.$$

The function $\Gamma(x)$ can be extended to a meromorphic function with poles at non-positive integers. For values outside of those poles, the gamma function satisfies the functional equation

$$\Gamma(x+1) = x\Gamma(x), \tag{3}$$

which can easily be derived using integration by parts. By (3), one has that for $n \in \mathbb{Z}_{\geq 0}$,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{4}$$

Theorem 1.1 (Euler's Reflection Formula, Theorem 1.2.1 of [1])**.** *For $a \in \mathbb{C}$ and $a \notin \mathbb{Z}$,*

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}.$$

It follows $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2\sqrt{3}\pi}{3}$.

The gamma function also satisfies multiplication formulas.

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Theorem 1.2 (Legendre's Duplication Formula, Theorem 1.5.1 of [1])**.** *For $a \in \mathbb{C}$,*

$$\Gamma(2a)\,(2\pi)^{1/2} = 2^{2a-\frac{1}{2}}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right). \tag{5}$$

Stated in terms of rising factorials, Theorem 1.2 gives that for all $n \in \mathbb{N}$,

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n. \tag{6}$$

Theorem 1.3 (Gauss' Multiplication Formula, Theorem 1.5.2 of [1])**.** *For $m \in \mathbb{N}$ and $a \in \mathbb{C}$,*

$$\Gamma(ma)(2\pi)^{(m-1)/2} = m^{ma-\frac{1}{2}}\Gamma(a)\Gamma\left(a+\frac{1}{m}\right)\cdots\Gamma\left(a+\frac{m-1}{m}\right).$$

Or equivalently, for any $n \in \mathbb{N}$,

$$(ma)_{mn} = m^{mn}\prod_{i=0}^{m-1}\left(a+\frac{i}{m}\right)_n.$$

Gamma values play important role in transcendental number theory. For example, Chudnovsky-Chudnovsky showed a few Gamma values including $\Gamma(\frac{1}{3})$ are transcendental. Meanwhile, p-adic Gamma values rationals are algebraic for a whole congruence class of primes p. For example, $\Gamma_p\left(\frac{1}{3}\right)$ is algebraic when $p \equiv 1 \mod 3$, see [7, Corollary 11.7.7] a textbook by Cohen.

Theorem 1.4 (Nesterenko [22])**.** *For any imaginary quadratic field with discriminant $-d$ and character $\epsilon(\cdot) = \left(\frac{-d}{\cdot}\right)$, the numbers*

$$\pi, \quad e^{\pi\sqrt{d}}, \quad \prod_{0<a<d}\Gamma\left(\frac{a}{d}\right)^{\epsilon(a)}$$

are algebraically independent.

Thus for example, $\Gamma\left(\frac{1}{3}\right), \pi, e^{\pi\sqrt{3}}$ are algebraically independent.

It is widely believed that (3), the reflection and multiplication formulas contain all algebraic relations among Gamma values for rational numbers. Its finite field analogue is called the Hasse conjecture, which was proved by Yamamoto **??**.

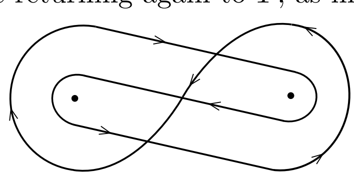
Definition 2. For $\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0$

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

The assumptions on x and y can be relaxed by integrating along the Pochhammer contour path around 0 and 1.

Definition 3. Let a, b be two points in $\mathbb{C}P^1$. Each Pochhammer contour γ_{ab} is a closed curve corresponding to a commutator of the form $ABA^{-1}B^{-1}$ in the fundamental group of $\pi_1(\mathbb{C}P^1 \setminus \{a,b,\infty\})$, where A, $B \in \pi_1(\mathbb{C}P^1 \setminus \{a,b,\infty\})$ are loops around both of the points a, b and the superscript -1 denotes a path taken in the opposite direction.

Example 1.1. For example, when $a = 0$, $b = 1$, a contour γ_{01} is a closed curve starting from a fixed point $T \in (0,1)$, going around 0 and 1 counterclockwise (in that order) and returning to T. Then one loops around 0 and 1 clockwise returning again to T, as indicated by the picture below.



Integrating over the double contour loop γ_{01}, the integral

$$B(x,y)=\frac{1}{(1-e^{2\pi i x})(1-e^{2\pi i y})}\int_{\gamma_{01}} t^{x-1}(1-t)^{y-1}\mathrm{d}t$$

converges for all values of x and y. For details, see [36].

Theorem 1.5 (Schneider [25])**.** *For any $a,b\in\mathbb{Q}$ such that $a,b,a+b\notin\mathbb{Z}$, then $B(a,b)$ is transcendental.*

So one of $\Gamma\left(\frac{1}{5}\right)$ and $\Gamma\left(\frac{2}{5}\right)$ is transcendental.

The power series expansion of a given analytic function $f(z)$ which is holomorphic at $z=0$ is a useful tool. The formula is given by

$$f(z)=\sum_{k\geq 0} f^{(k)}\frac{z^k}{k!}, \tag{7}$$

where $f^{(k)}$ stands for the kth derivative of f in terms of z. For example, given $a\in\mathbb{C}^\times$, for $f(z)=(1-z)^{-a}$, $f^{(k)}=-a(-a-1)\cdots(-a-k+1)(-1)^k=(a)_k$, so

$$(1-z)^{-a}=\sum_{k\geq 0}\frac{(a)_k}{k!}z^k. \tag{8}$$

The next result is generalization of the above.

Theorem 1.6 (Lagrange inversion theorem for formal power series)**.** *If $f(z)$ and $g(z)$ are formal power series where $g(0)=0$ and $g'(0)\neq 0$, then Lagrange's inversion theorem gives a way to write f as a power series in $g(z)$. In particular, one can write*

$$f(z)=f(0)+\sum_{k=1}^{\infty} c_k g(z)^k, \tag{9}$$

where

$$c_k=Res_z\frac{f'(z)}{kg(z)^k},$$

and $Res_z(f)$ denotes the coefficient of $1/z$ in the power series expansion of f.

1.2. **Hypergeometric data.** Let $\alpha=\{a_1,...,a_n\},\beta=\{1,b_2,...,b_n\}$ with $a_i,b_j\in\mathbb{Q}$ be a pair of multi-sets of the same length. Such a pair is called *primitive* if $a_i-b_j\notin\mathbb{Z}$ for any i,j. Let $M:=lcd(\alpha\cup\beta)$ the least positive common denominators of a_i,b_j's.

Definition 4. 1. A multiset $\alpha=\{a_1,...,a_n\}$ is called *defined* over $\mathbb{Q}$, if $\prod_{j=1}^n(X-e^{2\pi i a_j})\in\mathbb{Z}[X]$. I.e. for any integer c coprime to $lcd(\alpha)$, α and $c\alpha=\{ca_1,\cdots,ca_n\}\mod\mathbb{Z}$.

2. The pair α,β is said to be *self-dual* if it is congruent to the pair $-\alpha,-\beta$ mod $\mathbb{Z}$. A hypergeometric datum $HD=\{\alpha,\beta;\lambda\}$ is said to be *self-dual* if the pair α,β is self-dual; it is *defined over* $\mathbb{Q}$ if the pair α,β is defined over $\mathbb{Q}$ and $\lambda\in\mathbb{Q}^\times$.

1.3. **Classical hypergeometric functions.** The classical (generalized) *hypergeometric functions* ${}_nF_{n-1}$ with complex parameters $a_1,\ldots,a_n$, $b_1=1,b_2,\ldots,b_n$, and argument z are defined by

$${}_nF_{n-1}\left[\begin{matrix}a_1 & a_2 & \cdots & a_n\\ & b_2 & \cdots & b_n\end{matrix};\ z\right]:=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_n)_k}{(b_1)_k\cdots(b_n)_k}z^k=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_n)_k}{(b_2)_k\cdots(b_n)_k}\frac{z^k}{k!}, \tag{10}$$

and converge when $|z| < 1$. It also satisfies the following inductive integral relation of Euler [1, Equation (2.2.2)]. Namely, when $\mathrm{Re}(b_n) > \mathrm{Re}(a_{n+1}) > 0$,

$$(11)\quad {}_nF_{n-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_n \\ & b_2 & \cdots & b_n \end{matrix}\; ;\; z\right] = B(a_n, b_n - a_n)^{-1} \cdot \int_0^1 t^{a_n - 1}(1-t)^{b_n - a_n - 1} \cdot {}_{n-1}F_{n-2}\left[\begin{matrix} a_1 & a_2 & \cdots & a_{n-1} \\ & b_2 & \cdots & b_{n-1} \end{matrix}\; ;\; zt\right] \mathrm{d}t.$$

For the sake of a natural development in the finite field setting, we define in view of (8), for general $a \in \mathbb{C}$,

$$(12)\qquad {}_1P_0[a; z] := (1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = {}_1F_0[a; z].$$

Here we are using the notation ${}_1P_0$ to indicate a relationship to periods of algebraic varieties when $a \in \mathbb{Q}$. Next we let

$$(13)\quad {}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix}\; ; z\right] := \int_0^1 t^{b-1}(1-t)^{c-b-1}\, {}_1P_0\left[a; zt\right] \mathrm{d}t = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \mathrm{d}t,$$

or one can define it using the Pochhammer contour as in [27, §1.6]. To relate it to the ${}_2F_1$ function, one uses (11) to see that when $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$$(14)\qquad \begin{aligned} {}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix}\; ; z\right] &= \int_0^1 t^{b-1}(1-t)^{c-b-1} \cdot {}_1P_0[a; zt]\mathrm{d}t \\ &= \int_0^1 t^{b-1}(1-t)^{c-b-1} \cdot {}_1F_0[a; zt]\mathrm{d}t \\ &= B(b, c-b) \cdot {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\; ;\; z\right]. \end{aligned}$$

Inductively one can define the (higher) periods ${}_{n+1}P_n$ similarly by

$$(15)\quad {}_{n+1}P_n\left[\begin{matrix} a_1 & a_2 & \cdots & a_{n+1} \\ & b_2 & \cdots & b_{n+1} \end{matrix}\; ; z\right] := \int_0^1 t^{a_{n+1}-1}(1-t)^{b_{n+1}-a_{n+1}-1}\, {}_nP_{n-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_n \\ & b_2 & \cdots & b_n \end{matrix}\; ; zt\right] \mathrm{d}t.$$

In this formulation, the order of the a_i's (resp. b_j's) matters. Again using the beta function, one can show that when $\mathrm{Re}(b_i) > \mathrm{Re}(a_{i+1}) > 0$ for each $i \geq 1$,

$${}_{n+1}F_n\left[\begin{matrix} a_1 & a_2 & \cdots & a_{n+1} \\ & b_2 & \cdots & b_{n+1} \end{matrix}\; ;\; z\right] = \prod_{i=2}^{n+1} B(a_i, b_i - a_i)^{-1} \cdot {}_{n+1}P_n\left[\begin{matrix} a_1 & a_2 & \cdots & a_{n+1} \\ & b_2 & \cdots & b_{n+1} \end{matrix}\; ; z\right].$$

By the definition of ${}_{n+1}F_n$ in (10), any given ${}_{n+1}F_n$ function satisfies two nice properties:

1) The leading coefficient is 1;
2) The roles of the upper entries a_i's (resp. lower entries b_j's) are symmetric.

Clearly, the ${}_{n+1}P_n$ period functions do not satisfy these properties in general. The hypergeometric functions can thus be viewed as periods that are 'normalized' so that both properties 1) and 2) are satisfied.

Definition 5. A hypergeometric function with $\alpha = \{a_1, \cdots, a_n\}, \beta = \{1, b_2, \cdots, b_n\}$ is said to be *well-posed* if $a_i + b_i$ is a constant for all $i \in [1, n]$.

In [34] Whipple investigated well-posed series with argument ± 1. He obtained a few evaluation formulas and we will mention a few in this note.

Definition 6. A hypergeometric function $F(\alpha,\beta;x)$ with $\alpha=\{a_1,\cdots,a_n\},\beta=\{1,b_2,\cdots,b_n\}$ is said to be *k-balanced* where k is a positive integer, if the following conditions are satisfied

i). $x=1$
ii). one of the a_i's is a negative integer
iii). $k+\sum_{i=1}^n a_i=\sum_{i=2}^n b_i$.

When $k=1$, the series is called *balanced* or *Saalchützian.*

1.4. **Hypergeometric differential equations.** See [1, 35] for more details. Each ${}_nF_{n-1}$ function satisfies an order n ordinary Fuchsian differential equation in the variable λ with three regular singularities at 0, 1, and ∞ [27, §2.1.2][35].

Given $\alpha=\{a_1,\cdots,a_n\},\beta=\{1,b_2\cdots,b_n\}$, let $F(\alpha,\beta;\lambda)=\displaystyle\sum_{k\geq 0}A(k)\lambda^k$ where $A(k)=\displaystyle\prod_{i=1}^n\frac{(a_i)_k}{(b_i)_k}$.
Let $\theta_\lambda=\lambda\frac{d}{d\lambda}$. As $\theta_\lambda(fg)=(\theta_\lambda f)g+f(\theta_\lambda g)$, it is a derivation. In particular, $\theta_\lambda\lambda^k=k\lambda^k$. Then

$$(\theta_\lambda+c)(\sum_{k\geq 0}a_k\lambda^k)=\sum_{k\geq 0}(k+c)a_k\lambda^k.$$

Lemma 1.7. *Use D to denote $\frac{d}{d\lambda}$. For any integer $k\geq 1$,*

$$D^k=\frac{1}{\lambda^k}(\theta_\lambda-k+1)\cdots(\theta_\lambda-1)\theta_\lambda.$$

Proof. Prove by induction. When $k=1$, $Df=\frac{1}{\lambda}\theta_\lambda f$. Assume the claim holds for $k<n$. When $k=n$,

$$\begin{aligned}D(D^{n-1}f)&=\frac{1}{\lambda}\theta_\lambda\left(\frac{1}{\lambda^{n-1}}(\theta_\lambda-n+2)\cdots(\theta_\lambda-1)\theta_\lambda f\right)\\&=\frac{1}{\lambda}\left(\theta_\lambda(\frac{1}{\lambda^{n-1}})\right)(\theta_\lambda-n+2)\cdots(\theta_\lambda-1)\theta_\lambda f+\frac{1}{\lambda^n}\theta_\lambda(\theta_\lambda-n+2)\cdots(\theta_\lambda-1)\theta_\lambda f\\&=(1-n)\frac{1}{\lambda^n}(\theta_\lambda-n+2)\cdots(\theta_\lambda-1)\theta_\lambda f+\frac{1}{\lambda^n}\theta_\lambda(\theta_\lambda-n+2)\cdots(\theta_\lambda-1)\theta_\lambda f\\&=\frac{1}{\lambda^n}(\theta_\lambda-n+1)(\theta_\lambda-n+2)\cdots(\theta_\lambda-1)\theta_\lambda f\end{aligned}$$

□

Lemma 1.8. *For any $a\in\mathbb{C}$, $\lambda(\theta_\lambda+a)=(\theta_\lambda+a-1)\lambda$*

Proof. Right hand side applies to a function f yields

$$(\theta_\lambda+a-1)\lambda f=\lambda\frac{d\lambda f}{d\lambda}+(a-1)\lambda f=\lambda f+\lambda^2\frac{df}{d\lambda}+(a-1)\lambda f=\lambda^2\frac{df}{d\lambda}+a\lambda f=\lambda(\theta_\lambda+a)f.$$

□

Hence λ and $\theta_\lambda+a$ do not commute.

Lemma 1.9. *For any $a,b\in\mathbb{C}$, $\theta_\lambda+a$ and $\theta_\lambda+b$ commute.*

Proof. For constants a,b we have $(\theta_\lambda+a)(\theta_\lambda+b)=\theta_\lambda\theta_\lambda+(a+b)\theta_\lambda+ab$, which is symmetric in a and b. □

Lemma 1.10. *Given* $\alpha = \{a_1, \cdots, a_n\}, \beta = \{1, b_2 \cdots, b_n\}$, *let* $F(\alpha, \beta; \lambda) = \sum_{k\geq 0} A(k)\lambda^k$ *as before. Let*

$$\mathcal{L}_{\alpha,\beta;\lambda} := \prod_{i=1}^{n}(\theta_\lambda + b_i - 1) - \lambda \prod_{i=1}^{n}(\theta_\lambda + a_i), \tag{16}$$

then

$$\mathcal{L}_{\alpha,\beta;\lambda}\left(F(\alpha, \beta; \lambda)\right) = 0.$$

Proof. From the expression of $A(k)$ in terms of Pochhammer symbol, $A(k+1)/A(k) = \prod_{i=1}^{n} \frac{(a_i+k)}{(b_i+k)}$. Thus

$$\begin{aligned}\mathcal{L}_{\alpha,\beta;\lambda}F(\alpha, \beta; \lambda) &= \sum_{k\geq 0} k(k + b_2 - 1)\cdots(k + b_n - 1)A(k)\lambda^k - (k + a_1)\cdots(k + a_n)A(k)\lambda^{k+1} \\ &= \sum_{k\geq 0} \left((k + 1)\cdots(k + b_n)A(k + 1) - (k + a_1)\cdots(k + a_n)A(k)\right)\lambda^{k+1} = 0.\end{aligned}$$

□

1.5. **Indicial equations and characteristic exponents for singularities.** Each differential operator $\mathcal{L}_{\alpha,\beta;\lambda}$ is Fuchsian with only three regular singularities (see [2, Definition 2.2]) at 0, 1, ∞. For instance, ${}_2F_1\left[\begin{matrix} a_1 & a_2 \\ & b_2 \end{matrix}; \lambda\right]$ is a solution of the following hypergeometric differential equation $L_{\{a_1,a_2\},\{1,b_2\};\lambda}F = 0$ which is normalized so that the coefficient of D^2 is 1, where

$$\mathcal{L}_{\{a_1,a_2\},\{1,b_2\};\lambda} = D^2 + \frac{b_2 - (a_1 + a_2 + 1)\lambda}{\lambda(1 - \lambda)}D - \frac{a_1 a_2}{\lambda(1 - \lambda)}. \tag{17}$$

Now we recall how to define the local exponent of a normalized Fuchsian differential equation in variable x of the form $L = D^n + a_{n-1}(x)D^{n-1} + \cdots + a_0(x)$ where $a_i(x)$ are rational functions of x. Assume $x = 0$ is a singularity of L and a local solution is of the form $f = x^r(c_0 + c_1 x + x_2 x^2 + \cdots)$, where $c_0 \neq 0$. Thus

$$\begin{aligned} Df &= \sum_{k=0}^{\infty}(k + r)c_k x^{r+k-1}, \\ D^2 f &= \sum_{k=0}^{\infty}(k + r - 1)(k + r)c_k x^{r+k-2}, \\ &\cdots \\ D^n f &= \sum_{k=0}^{\infty}(k + r - n + 1)\cdots(k + r - 1)(k + r)c_k x^{r+k-n}. \end{aligned}$$

So $Lf = 0$ implies that necessarily the combined coefficient of the lowest non-trivial power of x, which is x^{r-n}, has to be 0. From the above computation we know the coefficient of x^{r-n} in Lf is c_0 times

$$r(r - 1)\cdots(r - n + 1) + C_{n-1}r(r - 1)\cdots(r - n + 2) + \cdots + C_0,$$

where $C_i = a_i(x)x^{n-i}|_{x=0}$. Setting the above to be 0 gives the so called *characteristic or indicial equation* of $Lf = 0$ near the singularity 0. The solutions of this degree-n polynomial of r are called the *characteristic exponents* of $Lf = 0$ near 0. Similarly, near any other singularity $a \neq \infty$, the indicial equation is

$$r(r - 1)\cdots(r - n + 1) + C_{n-1}r(r - 1)\cdots(r - n + 2) + \cdots + C_0 = 0, \tag{18}$$

where $C_i = a_i(x)(x-a)^{n-i}|_{x=a}$. If ∞ is a singularity, a similar formula is available when the local uniformizer is chosen as $1/x$, which is omitted here.

In other words, the local exponents of $\mathcal{L}_{\alpha,\beta;\lambda}$ can be understood as follows:

Let a be a singularity and w be the local parameter around a. The solutions around a take the form $w^r f(w)$, where $f(w)$ is locally holomorphic with $f(0) \neq 0$. The number r is called an exponent of the solution at a. If the local exponents $\{r_1, \dots, r_n\}$ of the differential operator at a are distinct, then a basis of local solutions near a can be given in the following form

$$w^{r_1} f_1(w), \cdots, w^{r_n} f_n(w)$$

with f_i holomorphic around a. If $e_i - e_j \in \mathbb{Z}$, an additional logarithmic term is allowed.

Example 1.2. For example, if we consider the differential equation (17), its indicial equation at singularity 0 is

$$r(r-1) + b_2 r = r(r-1+b_2) = 0,$$

thus the local exponent near 0 are $0, 1-b_2$. Near 1, the indicial equation is

$$r(r-1) + (-b_2 + a_1 + a_2 + 1)r = r(r - b_2 + a_1 + a_2) = 0,$$

so the local exponents near 0 are $0, b_2 - a_1 - a_2$.

Theorem 1.11. *The local exponents of* $\mathcal{L}_{\alpha,\beta;\lambda}$ *are*

$$\begin{aligned} 0, 1-b_2, \cdots, 1-b_n \quad & \textit{at } \lambda = 0 \\ a_1,\, a_2,\, \cdots,\, a_n \quad & \textit{at } \lambda = \infty \\ 0, 1, 2, \cdots, n-2, \gamma \quad & \textit{at } \lambda = 1, \end{aligned} \tag{19}$$

where

$$\gamma = -1 + \sum_{j=1}^{n} b_j - \sum_{j=1}^{n} a_j \tag{20}$$

See [2, §2] by Beukers and Heckman.

Information of the singular points and their characteristic exponents are often arranged in columns as below, which is called the *Riemann scheme* of the differential equation. In the last column, we indicate which the variable is in used. The Riemann scheme for $\mathcal{L}_{\{a_1,a_2\},\{1,b_2\};x}$ is as follows:

$$P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & a_1 & ; & x \\ 1-b_2 & b_2 - a_1 - a_2 & a_2 & & \end{pmatrix},$$

which is used to denote the solutions set of $\mathcal{L}_{\{a_1,a_2\},\{1,b_2\};x}$.

Note that linear fractional transformations permuting $0, 1, \infty$ are

$$x \mapsto x, 1-x, \frac{1}{x}, \frac{1}{1-x}, \frac{x-1}{x}, \frac{x}{x-1}.$$

When $x \mapsto 1-x$, we have

$$P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & a_1 & ; & x \\ 1-b_2 & b_2 - a_1 - a_2 & a_2 & & \end{pmatrix} = P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & a_1 & ; & 1-x \\ b_2 - a_1 - a_2 & 1-b_2 & a_2 & & \end{pmatrix}.$$

Similarly

$$(21)\quad P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & a_1 & ; & x \\ 1-b_2 & b_2-a_1-a_2 & a_2 & & \end{pmatrix} =$$

$$P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & a_1 & 0 & ; & \frac{x}{x-1} \\ 1-b_2 & a_2 & b_2-a_1-a_2 & & \end{pmatrix} = (1-x)^{-a_1} P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & a_1 & ; & \frac{x}{x-1} \\ 1-b_2 & a_2-a_1 & b_2-a_2 & & \end{pmatrix}.$$

In the last step, when $(1-x)^{-a_1}$ is pulled out, it only affects the local exponents at 1 and ∞.

Exercise 1.1. Check that the Riemann Schemes for

$${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right], \quad \text{and } z^{1-c}(1-z)^{c-a-b}\,{}_2F_1\left[\begin{matrix} 1-a & 1-b \\ & 2-c \end{matrix}; z\right]$$

are the same. So they satisfy the same differential operator $\mathcal{L}_{\{a,b\},\{1,c\},z}$. If $c \neq 1$, they are not scalar multiple of each other.

1.6. **Local solutions.** Next we consider local solutions to $\mathcal{L}_{\{a_1,a_2\},\{1,b_2\};z}$ near the singularities.

(1) Around the singularity $z = 0$, if $b_2 \notin \mathbb{Z}$, a basis of the solution space can be given by

$$f_0 = {}_2F_1\left[\begin{matrix} a_1 & a_2 \\ & b_2 \end{matrix}; z\right]$$
$$g_0 = (z)^{1-b_2}\,{}_2F_1\left[\begin{matrix} 1+a_1-b_2 & 1+a_2-b_2 \\ & 2-b_2 \end{matrix}; z\right].$$

When $b_2 \in \mathbb{Z}$, we consider

$$f_0(c) := {}_2F_1\left[\begin{matrix} a_1 & a_2 \\ & c \end{matrix}; z\right]$$
$$g_0(c) := (z)^{1-c}\,{}_2F_1\left[\begin{matrix} 1+a_1-b_2 & 1+a_2-c \\ & 2-c \end{matrix}; z\right]$$

as functions of c. Then a basis of the solution space can be taken by f_0 and

$$h_0 = \lim_{c\to b_2} \frac{g_0 - f_0}{c - b_2}.$$

For example, when $b_2 = 1$, f_0 and g_0 are two linearly independent solutions when c near 1. The function h_0

$$\begin{aligned} h_0 &= \lim_{c\to 1} \frac{g_0 - f_0}{c-1} \\ &= \frac{d}{dc} z^{1-c} F(\{1+a_1-c, 1+a_2-c\}, \{2-c\}; z) \,|_{c=1} \\ &= \log z \cdot f_0 + \sum_{m=1}^{\infty} \frac{(a_1)_m (a_2)_m}{m!^2} z^m \left[\sum_{k=1}^{m} \left(\frac{1}{a_1+k-1} + \frac{1}{a_2+k-1} - \frac{2}{k}\right)\right] \end{aligned}$$

gives a solution, which is linearly independent to f_0.

(2) Around the singularity $z=1$, if $a_1+a_2-b_2\notin\mathbb{Z}$, a basis of the solution space can be taken by

$$f_1 := {}_2F_1\left[\begin{matrix} a_1 & a_2 \\ & 1+a_1+a_2-b_2 \end{matrix}; 1-z\right]$$
$$g_1 := (1-z)^{b_2-a_1-a_2}\,{}_2F_1\left[\begin{matrix} b_2-a_1 & b_2-a_2 \\ & 1+b_2-a_1-a_1 \end{matrix}; 1-z\right].$$

To obtain such a basis, one can consider the differential equation under the transformation $z\mapsto 1-z$. This converts the differential equation given by (17) to the equation

$$\left(D^2+\frac{a_1+a_2+1-b_2-(a_1+a_2+1)z}{z(1-z)}D-\frac{a_1a_2}{z(1-z)}\right)F=0$$

(3) Around $z=\infty$, and $a_1-a_2\notin\mathbb{Z}$, a basis of the solution space can be taken by

$$f_\infty := z^{-a_1}\,{}_2F_1\left[\begin{matrix} a_1 & 1+a_1-b_2 \\ & 1+a_1-a_2 \end{matrix}; 1/z\right]$$
$$g_\infty := z^{-a_2}\,{}_2F_1\left[\begin{matrix} a_2 & 1+a_2-b_2 \\ & 1+a_2-a_1 \end{matrix}; 1/z\right].$$

Similarly, we can set $z=1/t$ and transform the original equation

$$\left((a_1+\theta_z)(a_2+\theta_z)-(1+\theta_z)(b_2+\theta_z)\frac{1}{z}\right)F=0$$

to

$$\left((a_1-\theta_t)(a_2-\theta_t)-(1-\theta_t)(b_2-\theta_t)t\right)F=0.$$

By the fact that

$$\begin{aligned}&\left((a_1-\theta_t)(a_2-\theta_t)-(1-\theta_t)(b_2-\theta_t)t\right)t^a\\ =&t^{a+1}\left((\theta_t+a+1-b_2)(\theta_t+a)-(\theta_t+a+1-a_1)(\theta_t+a+1-a_a)\frac{1}{t}\right),\end{aligned}$$

we can choose

$$t^{a_1}\,{}_2F_1\left[\begin{matrix} a_1 & 1+a_1-b_2 \\ & a_1-a_2+1 \end{matrix}; t\right]$$
$$t^{a_2}\,{}_2F_1\left[\begin{matrix} a_2 & 1+a_a-b_2 \\ & a_2-a_1+1 \end{matrix}; t\right]$$

as a basis for the solution space around $t=0$, i.e, $z=\infty$.

For general hypergeometric differential equations, the local solutions spaces can be described in a similar way.

When no b_j, $j=2,\cdots,n$ is an integer, and no two b_j differ by an integer, a fundamental set of solutions of $\mathcal{L}_{\alpha,\beta;\lambda}F=0$ is given by

$$\begin{aligned}&F(\alpha,\beta;z)\\ &z^{1-b_k}F(\{1+a_i-b_k\},\{1+b_j-b_k\};z),\quad k=2,\cdots,n.\end{aligned}$$

When no two a_i differ by an integer, a fundamental set of solutions around $z=\infty$ is given by

$$f_{\infty,k}(z) := (-z)^{-a_k}F(\{1-b_j+a_k\},\{1-a_i+a_k\};\frac{1}{z}),\quad k=1..n,$$

and the relation between the solutions are

$$F(\alpha,\beta;z)=\sum_{k=1}^{n}\left(\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{\Gamma(a_j-a_k)}{\Gamma(a_j)}\right)\left(\prod_{j=1}^{n}\frac{\Gamma(b_j)}{\Gamma(b_j-a_k)}\right)f_{\infty,k}(z).$$

See [10, (16.8.8)].

If one of $b_j,\ j=2,\cdots,n$ is an integer, one can find solutions through the continuity of the hypergeometric functions as functions of the parameters b_j.

1.7. **Monodromy representations and Beukers-Heckman theorem.** Please see [2] for full details.

Given an order n ordinary Fuchsian differential equation L with only regular singularities $x_1,\cdots,x_s$, we consider the monodromy representation of the fundamental group $\pi_1(\mathbb{C}P^1\setminus\{x_1,\cdots,x_s\},x_0)$ where x_0 is any ordinary point as follows. By Cauchy, near x_0, the solution space of the homogeneous equation $Lu=0$ is an n-dimensional vector space $V(x_0)$ over $\mathbb{C}$. We fix a basis $f_1,\cdots,f_n$ of $V(x_0)$. Let $L(t):[0,1]\to\mathbb{C}P^1\setminus\{x_1,\cdots,x_s\}$ be a continuous function with $L(0)=L(1)=x_0$. Its image is topologically a closed loop. We extend $f_1,\cdots,f_n$ analytically along L when t varies from 0 to 1. By Frobenius' result, $f_1(L(1)),\cdots,f_n(L(1))$ form another basis of $V(x_0)$. Thus it can be written as $M(L)(f_1,\cdots,f_n)^T$ where $M(L)\in GL_n(\mathbb{C})$ only depending on the class $[L]$ of L in $\pi_1(\mathbb{C}P^1\setminus\{x_1,\cdots,x_s\},x_0)$. The map from $[L]\mapsto M(L)$ is a homomorphism, which is unique up to conjugation by $GL_n(\mathbb{C})$. It is called a *monodromy representation* of L.

Let $\mathfrak{a}_j=e^{2\pi i a_j},\mathfrak{b}_j=e^{2\pi i b_j}$. They are roots of unity lie on the unit circle in $\mathbb{C}$, under our assumptions on $\alpha=\{a_1,\cdots,a_n\}$ and $\beta=\{b_1,\cdots,b_n\}$ with $a_i,b_j\in\mathbb{Q}$.

Definition 7. Suppose $0\le a_1\le\cdots a_n<1, 0\le b_1\le\cdots b_n\le 1$. We say the sets $\mathfrak{a}_1=e^{2\pi i a_1},\cdots,\mathfrak{a}_n=e^{2\pi i a_n}$ and $\mathfrak{b}_1=e^{2\pi i b_1},\cdots,\mathfrak{b}_n=e^{2\pi i b_n}$ *interlace* on the unit circle if and only if either

$$a_1<b_1<a_2<b_2<\cdots<a_n<b_n \quad\text{or}\quad b_1<a_1<\cdots<b_n<a_n.$$

If α,β is a primitive pair, then the monodromy group is an irreducible subgroup of $GL_n(\mathbb{C})$, local monodromy matrix M_1 near 1 is a reflection (i.e. M_1-id has rank 1), with determinant $c=e^{2\pi i\gamma}$, where $\gamma=-1+\sum_{j=1}^n b_j-\sum_{j=1}^n a_j$ in equation (20).

Theorem 1.12 (Levelt, see Theorem 3.5 [2])**.** *Let A_j,B_k be defined by*

$$\prod_{j=1}^{n}(X-\mathfrak{a}_j)=X^n+A_1X^{n-1}+\cdots+A_n,\quad \prod_{j=1}^{n}(X-\mathfrak{b}_j)=X^n+B_1X^{n-1}+\cdots+B_n, \tag{22}$$

then

$$M_\infty=\begin{pmatrix}0&0&\cdots&0&-A_n\\1&0&\cdots&0&-A_{n-1}\\0&1&\cdots&0&-A_{n-2}\\&&\cdots&&\\0&0&\cdots&1&-A_1\end{pmatrix},\quad M_0=\begin{pmatrix}0&0&\cdots&0&-B_n\\1&0&\cdots&0&-B_{n-1}\\0&1&\cdots&0&-B_{n-2}\\&&\cdots&&\\0&0&\cdots&1&-B_1\end{pmatrix}^{-1},\quad M_1=M_\infty^{-1}M_0^{-1}$$

generate a hypergeometric group which is conjugate inside $GL_n(\mathbb{C})$ to the monodromy group of the hypergeometric differential equation with parameters α and β.

Denote this group by $H(\{\mathfrak{a}_1,\cdots,\mathfrak{a}_n\},\{\mathfrak{b}_1,\cdots,\mathfrak{b}_n\})$. When α and β form a primitive pair, then $H(\{\mathfrak{a}_1,\cdots,\mathfrak{a}_n\},\{\mathfrak{b}_1,\cdots,\mathfrak{b}_n\})$ is irreducible.

Remark 1. *Due to a result of Pochhamer (see Proposition 2.8* [2]*), the local monodromy matrix* M_1 *is called a (quasi)reflection as it satisfies a special property that* $M_1 - nI_n$ *has only rank 1, where* I_n *stands for the rank-n identity matrix.*

Example 1.3. For $\alpha = \{\frac{1}{2}, \frac{1}{2}\}, \beta = \{1, 1\}$, the two polynomials are $X^2 + 2X + 1$ and $X^2 - 2X + 1$, so $H_{\{-1,-1\},\{1,1\}} = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \right\rangle$. Note that

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} H_{\{-1,-1\},\{1,1\}} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^{-1} = \Gamma(2)$$

the principal level-2 congruence subgroup.

The above theorem of Levelt is a rigidity theorem. When $n = 2$, see Theorem 2.3.1 of [1] due to Papperitz. Equivalently, it can be stated as

Theorem 1.13. *Every second Fuchsian equation with 3 regular singularities can be transformed to a hypereometric differential equation.*

More generally,

Theorem 1.14 (Rigidity Theorem)**.** *Each order-n ordinary differential equation in variable z which has only three regular singularities at 0, 1,* ∞ *and the corresponding indicial exponents as* (19) *is equivalent to* $\mathcal{L}_{\alpha,\beta;z}F = 0$.

Exercise 1.2. Assume $a \notin \mathbb{Z}$. Show that ${}_2F_1\left[\begin{matrix} a & 1-a \\ & 1 \end{matrix}; x\right]$ and ${}_2F_1\left[\begin{matrix} a & 1-a \\ & 1 \end{matrix}; 1-x\right]$ are two linear independent solutions of $\mathcal{L}_{\{a,1-a\},\{1,1\};x}$.

In [2] Beukers and Heckman gave explicit descriptions for when $H(\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}, \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\})$ is finite.

Theorem 1.15 (Beukers and Heckman)**.** *Let* $M = lcd(\alpha \cup \beta)$ *be the least positive common denominators of* a_i, b_j*'s. The hypergeometric group* $H(\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}; \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\})$ *is finite if and only if for each* $k \in \mathbb{N}$ *coprime to* M*, the sets* $\{\mathfrak{a}_1^k, \cdots, \mathfrak{a}_n^k\}$ *and* $\{\mathfrak{b}_1^k, \cdots, \mathfrak{b}_n^k\}$ *interlace on the unit circle.*

Remark 2. *When both* α, β *are defined over* $\mathbb{Q}$*, we only need to check whether* $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}$ *and* $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\}$ *interlace.*

Example 1.4. According to the above theorem, the hypergeometric group for the multi-sets $\alpha = \{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}\}, \beta = \{1, \frac{1}{3}, \frac{2}{3}\}$ is finite; while hypergeometric group for the multi-sets $\alpha = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}, \beta = \{1, \frac{1}{6}, \frac{5}{6}\}$ is not.

We plot the positions of $\mathfrak{a}_1, \cdots, \mathfrak{a}_3$ (in green color) and $\mathfrak{b}_1, \cdots, \mathfrak{b}_3$ (in red color) on the unit circle for these two cases respectively.

When the hypergeometric group is infinite, they have the following result.

Theorem 1.16 (Beukers and Heckman)**.** *Let* $H(\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}, \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\})$ *be an infinite primitive hypergeometric group which is not a scalar shift of a finite group. Let* $\overline{H}(\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}, \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\})$ *be its Zariski closure. Then there are two possibilities,*

I) There exists $d \in \mathbb{C}^\times$ *such that*

$$\{d\mathfrak{a}_1, \cdots, d\mathfrak{a}_n\} = \{(d\mathfrak{a}_1)^{-1}, \cdots, (d\mathfrak{a}_n)^{-1}\}, \quad \text{and } \{d\mathfrak{b}_1, \cdots, d\mathfrak{b}_n\} = \{(d\mathfrak{b}_1)^{-1}, \cdots, (d\mathfrak{b}_n)^{-1}\}.$$

Ia) If $c = e^{2\pi i\gamma} = 1$ *where* γ *as in* (20)*, then*

$$\overline{H}(\{d\mathfrak{a}_1, \cdots, d\mathfrak{a}_n\}, \{d\mathfrak{b}_1, \cdots, d\mathfrak{b}_n\}) = Sp_n(\mathbb{C});$$

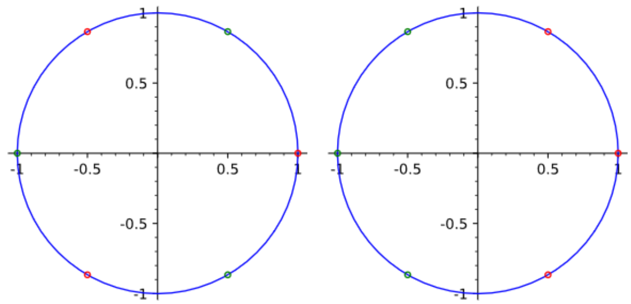


FIGURE 1. $\{\alpha = \{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}\}, \beta = \{1, \frac{1}{3}, \frac{2}{3}\}\}$ vs $\{\alpha = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}, \beta = \{1, \frac{1}{6}, \frac{5}{6}\}\}$

Ib) if $c = -1$ *then*

$$\overline{H}(\{d\mathfrak{a}_1, \cdots, d\mathfrak{a}_n\}, \{d\mathfrak{b}_1, \cdots, d\mathfrak{b}_n\}) = O_n(\mathbb{C}).$$

II) For the remaining cases, $SL_n(\mathbb{C}) \subset \overline{H}(\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}, \{\mathfrak{b}_1, \cdots, \mathfrak{b}_n\})$.

Example 1.5. For $\alpha = \{\frac{1}{2}, \frac{1}{2}\}, \beta = \{1, 1\}$ as in Example 1.3, $d = 1$ and $c = 1$. In this case $\overline{H}(\{-1, -1\}; \{1, 1\}) = Sp_2(\mathbb{C})$. For $\alpha = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}, \beta = \{1, \frac{1}{6}, \frac{5}{6}\}$, $d = 1, c = -1$, the Zariski closure of its monodromy group is isomorphic to $O_3(\mathbb{C})$.

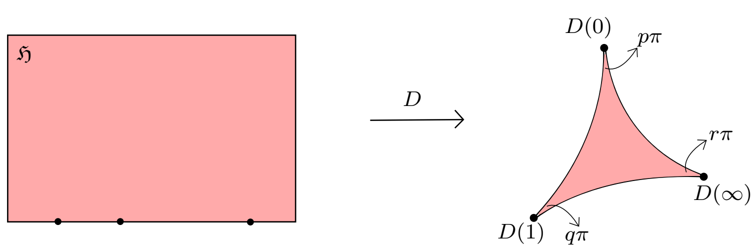
1.8. **Schwarz theorem, triangle groups and arithmetic triangle groups.** When $\alpha = \{a, b\}, \beta = \{1, c\}$, the following results are useful. There is an explicit correspondence between a hypergeometric differential equation $\mathcal{L}_{\alpha,\beta}$ and a Schwarz triangle $\Delta(p, q, r)$ with $p, q, r \in \mathbb{Q}$ due to the following theorem of Schwarz. See [3] by Beukers and [36] by Yoshida for more details.

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the complex upper half-plane and $\mathbb{P}^1 = \widehat{\mathbb{C}}$ or $\mathbb{CP}^1$.

Theorem 1.17 (Schwarz, (see [35]))**.** *Fix a* $z_0 \in \mathbb{H}$*, let* f, g *be two independent solutions to the differential equation* $\mathcal{L}_{\{a,b\},\{1,c\};z}F = 0$ *near at* z_0*, and let* $p = |1-c|$*,* $q = |c-a-b|$*, and* $r = |a-b|$*. If* p*,* q*,* $r < 1$*, then the Schwarz map*

$$D : \quad \mathbb{H} \cup \mathbb{R} \longrightarrow \mathbb{P}^1, \quad D(z) = f(z)/g(z)$$

gives a bijection from $\mathbb{H} \cup \mathbb{R}$ *onto a curvilinear triangle with vertices* $D(0)$*,* $D(1)$*,* $D(\infty)$ *and corresponding angles* $p\pi$*,* $q\pi$*,* $r\pi$*, as illustrated below.*



The pictures above are from [12].

Remark 3. *(1) By Cauchy's fundamental theorem, the solutions* f *and* g *do not vanish simultaneously (at almost all points). In addition, the map* $D(z)$ *is locally bijection for any* $z \in \mathbb{H}$*.*

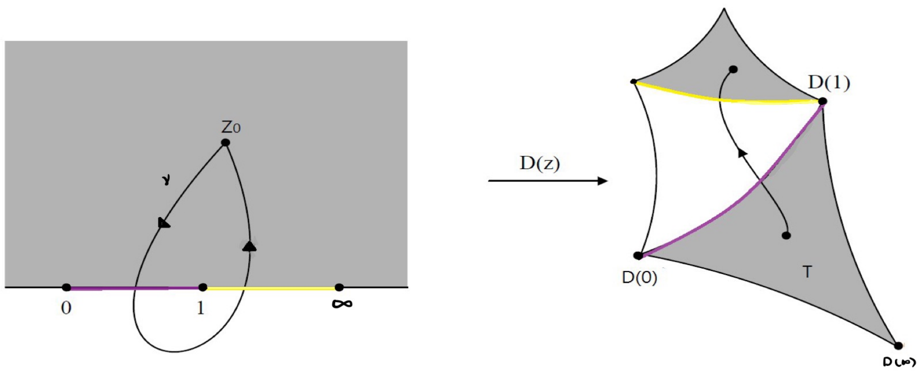
(2) A change of the basis $\{f, g\}$ *corresponds to a fractional linear transformation which does not change the angles of the curvilinear triangle.*

(3) *The image of the map $D(z)$ at $z = 1$ can be determined by the Gauss evaluation formula, see 29 below.*

Note that a Schwarz triangle with angles $p\pi$, $q\pi$, and $r\pi$ as described in Theorem 1.17 can be used to tile the sphere ($\mathbb{P}^1$), the Euclidean plane ($\mathbb{C}$), or the hyperbolic plane ($\mathbb{H}$) through reflections along its edges, depending on whether $p+q+r$ is equal to, greater than, or less than 1, respectively. Therefore, each Schwarz triangle $\Delta(p, q, r)$ can be associated to the symmetry group of this tiling, which we denote by $S_\Delta(p, q, r)$.

From the Schwarz' reflection principle [36, Proposition 7.1 and Corollary 7.2], we will see that every element of the projective monodromy group is a product of an even number of reflections in the edges of the curvilinear triangle. The idea is as follows: Let D be a Schwarz map.

(1) For a fixed $z_0 \in \mathbb{H}$, take a loop γ starting at z_0 passing the interval $(0, 1)$ into the lower half-plane $\overline{\mathbb{H}} := \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$ and returning to z_0 through $(1, \infty)$. Let ℓ_∞ be the edge connecting $D(0)$ and $D(1)$ and ℓ_0 be the edge connecting $D(\infty)$ and $D(1)$. The image of γ is a path in $\mathbb{P}^1$ starting at $D(z_0)$ in the triangle T, passing through ℓ_∞ into the mirror image of T, say T_∞, and then passing through ℓ_0 into the mirror image of T_∞.



(Source: Beukers' Note [3])

(2) Then new Schwarz map $\gamma \cdot D$ (analytic continuation along γ) is

$$\gamma \cdot D = \frac{aD + b}{cD + d}, \quad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$

In the same vein, $D(z)$ can be extended to $\overline{\mathbb{H}}$ through any interval ℓ of $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$ by

$$D(z) := \overline{D(\overline{z})}.$$

Then the extended D is holomorphic on $\mathbb{H} \cup \ell \cup \overline{\mathbb{H}}$ and $\mathrm{Image}(D) = D(H) \cup D(\ell) \cup D(\overline{\mathbb{H}})$. If $D(\ell)$ is part of a circle C, then $D(\overline{\mathbb{H}})$ is the mirror image of $D(\mathbb{H})$ with respect to C. That is,

$$D(\overline{\mathbb{H}}) = g^{-1} \cdot \left(\overline{g \cdot D(\mathbb{H})}\right),$$

where $g \in \mathrm{GL}_2(\mathbb{C})$ and $g(C) = \mathbb{R}$. If we make analytic continuations from $\mathbb{H}$ to $\overline{\mathbb{H}}$ and from $\overline{\mathbb{H}}$ to $\mathbb{H}$ (i.e. make an even number of reflections), we get a linear fractional transformation. Such transformations form the projective monodromy group Γ.

Let S be either $\mathbb{H}$, $\mathbb{C}$, or $\mathbb{P}^1$ equipped with the hyperbolic, euclidean and spherical metric respectively. For a fixed geodesic triangle Δ, denote

$W(\Delta) :=$ the group generated by the reflections in the edges of Δ.

Theorem 1.18 ([3])**.** *For any geodesic triangle Δ, we have $S = \bigcup_{\gamma \in W(\Delta)} \gamma \cdot \overline{\Delta}$, where $\overline{\Delta}$ is the closure of Δ in S.*

The picture below shows $\mathbb{H}$ can be tessellated by the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. In general, the (open) triangles may overlap, so we consider special triangles, called elementary

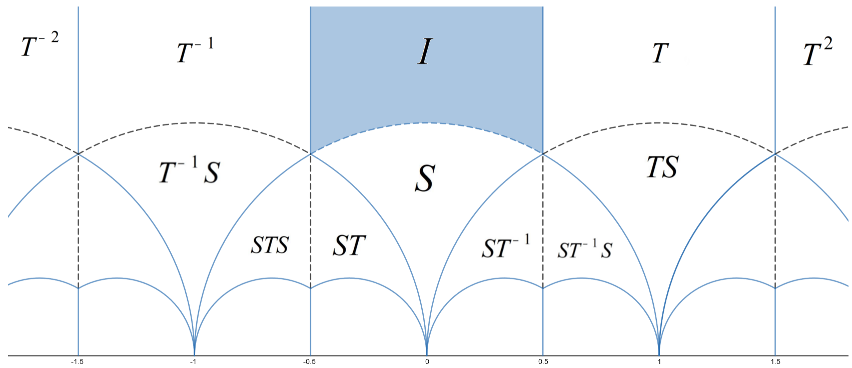


FIGURE 2. A partial tessellation of $\mathbb{H}$, photo courtesy of Bao Pham

triangles which give tessellation of S as well.

Definition 8. An *elementary triangle* is a geodesic triangle whose vertex angles are all of the form π/n, $n = 2, \ldots, \infty$.

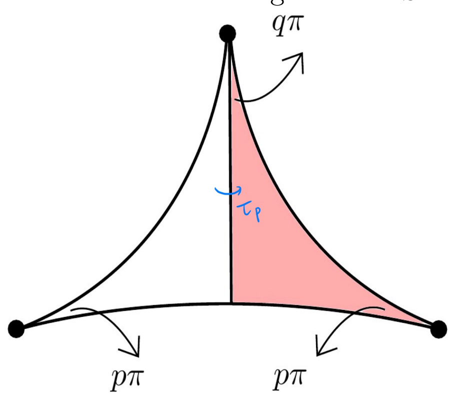
Theorem 1.19 ([3])**.** *Let Δ be an elementary triangle. Then, for any $\gamma \in W(\Delta)$, $\gamma \neq \pm 1$, we have*

$$\gamma\Delta \cap \Delta = \emptyset.$$

For any geodesic triangle Δ, if $W(\Delta)$ acts on S discretely, we can find an elementary triangle Δ_{et} such that $W(\Delta_{et}) = W(\Delta)$.

For example, for the triangle Δ with internal angles $(2\pi/5,\ 2\pi/5,\ 2\pi/5)$, the associated Δ_{et} has angles $(\pi/2, \pi/3, \pi/5)$.

We now assume that $|1-c| = p = 1/e_1$, $|c-a-b| = q = 1/e_2$, and $|a-b| = r = 1/e_3$ with $e_i = 2, \ldots, \infty$. Let τ_p, τ_q, τ_r be reflections in the edges of the Schwarz triangle $\Delta(p, q, r)$.



Set $g_p = \tau_q\tau_r$, $g_q = \tau_p\tau_r$, $g_r = \tau_p\tau_q$. Then the triangle group

$$\Gamma = (e_1, e_2, e_3) := \langle g_p, g_q, g_r : g_p^{e_1} = g_q^{e_2} = g_r^{e_3} = g_p g_q g_r = 1\rangle$$

is a subgroup of $\mathrm{Isom}^+(S)$. The quotient space $X(\Gamma) := \Gamma\backslash S$ is a Riemann orbifold of genus zero with $\#\{\infty \in \{e_1, e_2, e_3\}\}$ punctures.

There are three categories of triangle groups:

(1) $p+q+r>1$, there are only four types:

$$(2,2,n),\quad (2,3,3),\quad (2,3,4),\quad (2,3,5).$$

The corresponding hypergeometric functions are algebraic. For instance,

$${}_2F_1\left[\begin{matrix} a & \frac{1}{2}+a \\ & \frac{1}{2} \end{matrix}\,;\, z\right] = \frac{1}{2}\left((1+\sqrt{z})^{-2a} + (1-\sqrt{z})^{-2a}\right).$$

(2) $p+q+r=1$, there are only four cases:

$$(2,2,\infty),\quad (2,3,6),\quad (2,4,4),\quad (3,3,3).$$

(3) $p+q+r<1$, there are infinitely many cases.

Example 1.6.

$$(2,3,\infty)\simeq \mathrm{PSL}_2(\mathbb{Z}),\quad (\infty,\infty,\infty)\simeq\Gamma(2),\quad (2,3,3)\simeq A_4.$$

When $p+q+r<1$, we say that $\Gamma=(e_1,e_2,e_3)$ is *arithmetic* if Γ is commensurable with norm 1 group of certain quaternion order. There are exactly 85 arithmetic triangle groups falling into 19 classes (see the works of Takeuchi [30, 29]). The compactifications of their corresponding quotient spaces $X(\Gamma)$ are Shimura curves, which parametrize certain abelian surfaces with quaternionic multiplication. In the work of [30], Takeuchi gives the precise quaternion algebras and orders. From which one can realize the arithmetic triangle groups as subgroups of $\mathrm{SL}_2(\mathbb{R})$. Alternatively, there is an explicit embedding given by Petersson [23, 6] (we will describe this embedding in a later discussion).

Example 1.7 ([28]). Take $(2,3,\infty)\simeq\mathrm{PSL}_2(\mathbb{Z})$ for example. Let $J=j/1728$ be the hauptmodul for $\mathrm{PSL}_2(\mathbb{Z})$ such that $J(i)=1$, $J(\rho)=0$, and has a pole at cusp $i\infty$, where $\rho=e^{2\pi i/6}$.

Denote

$$f(J)=J^{-1/6}(1-J)^{1/4}\,{}_2F_1\left[\begin{matrix}\frac{1}{12} & \frac{1}{12}\\ & \frac{2}{3}\end{matrix}\,;\,J\right],\quad g(J)=J^{1/6}(1-J)^{1/4}\,{}_2F_1\left[\begin{matrix}\frac{5}{12} & \frac{5}{12}\\ & \frac{4}{3}\end{matrix}\,;\,J\right],$$

and define

$$D(J)=\frac{\rho f(J)+Cg(J)}{f(J)-C(2+\sqrt{3})g(J)},$$

where

$$C=\frac{1}{2}\frac{i-\rho}{i-\bar\rho}\cdot\zeta_{12}\cdot\frac{{}_2F_1\left[\begin{matrix}\frac{1}{12} & \frac{1}{12}\\ & \frac{2}{3}\end{matrix}\,;\,1\right]}{{}_2F_1\left[\begin{matrix}\frac{5}{12} & \frac{5}{12}\\ & \frac{4}{3}\end{matrix}\,;\,1\right]}=\frac{2-\sqrt{3}}{2}i\cdot\frac{\Gamma\left(\frac{11}{12}\right)^2\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{7}{12}\right)^2\Gamma\left(\frac{4}{3}\right)}.$$

Slit the disk $|J|<1$ along the negative real axis and use the principal branch of $\log J$ to define $J^{\pm1/6}$ so that it takes positive real values on $(0,1)$. In a neighborhood of $J=0$, $|J|<1$, the function D maps $(0,1)$ to the geodesic running from i to ρ. If we continue slitting the J-plane along the negative real axis and along the positive real axis from 1 to ∞, we can continue $D(J)$ to a single-valued function which maps the slit plane onto the fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$ in Figure 3.

Similarly, we can find a suitable basis to describe the Schwarz map as an inverse to the J-function through the standard fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$. Set $t=1/J$ and denote

$$f(t)=(1-t)^{1/4}\,{}_2F_1\left[\begin{matrix}\frac{1}{12} & \frac{5}{12}\\ & 1\end{matrix}\,;\,t\right],\quad g(t)=f(t)\left(\frac{1}{2\pi i}\log t+h(t)\right),$$

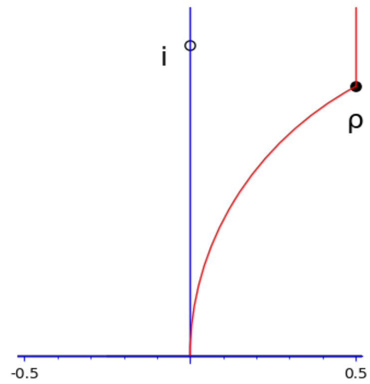


FIGURE 3. A fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$, photo courtesy of Bao Pham

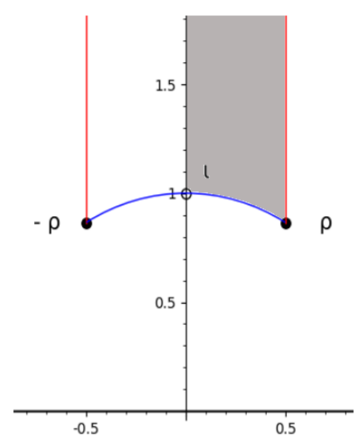
where $h(t)$ is a holomorphic function and vanishes at $t = 0$. We now choose the Schwarz map to be

$$D(t) = \frac{g(t)}{f(t)} - \frac{1}{2\pi i}\log 1728.$$

Then

$$\lim_{t\to 0} D(t) = \infty, \quad \lim_{t\to 1} D(t) = i,$$

the function D maps $(0, 1)$ to the geodesic running from $i\infty$ to i. This Schwarz map $D(t)$ is an inverse in a neighborhood of $t = 0$ to the J-function on $\mathbb{H}$.



1.9. **The Legendre curves.**

Theorem 1.20 ([36, 28])**.** *The Schwarz map*

$$D(\lambda) := i\frac{{}_2F_1\left[\begin{matrix}\frac12 & \frac12\\ & 1\end{matrix};\ 1-\lambda\right]}{{}_2F_1\left[\begin{matrix}\frac12 & \frac12\\ & 1\end{matrix};\ \lambda\right]}$$

of $L_{\{1/2,1/2\};\{1,1\}}$ *gives an isomorphism*

$$\mathbb{P}^1 - \{0,1,\infty\} \longrightarrow \Gamma(2)\backslash\mathbb{H},$$

which is the inverse map of the modular λ*-function, which parameterizes isomorphism classes of elliptic curves with level-2 structures.*

Similarly, we can choose a suitable basis $\{f,g\}$ of the $L_{\{1/12,5/12\};\{1,1\}}$ so that the corresponding Schwarz map gives an isomorphism

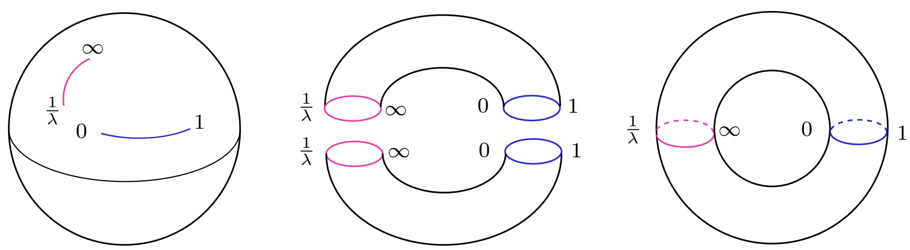
$$\mathbb{P}^1 - \{\infty\} \longrightarrow \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H},$$

which is the inverse map of the elliptic j-function.

We now elaborate some discussion behind Theorem 1.20 in which $\alpha = \{\frac{1}{2},\frac{1}{2}\}, \beta = \{1,1\}$. It is well-known that elliptic curves with level-2 structures can be written as the Legendre curves

$$L_\lambda : y^2 = x(1-x)(1-\lambda x). \tag{23}$$

When $\lambda \neq 0,1$ it is an elliptic curve. It is a double over of $\mathbb{C}P^1$ which ramifies only at $P_1 = 0, P_2 = 1, P_3 = \frac{1}{\lambda}, P_4 = \infty$ as demonstrated by the picture below. Going from right to left, first cut the torus twice including half of each of the indicated boundaries on each torus, to get two cylinders. Each cylinder can be realized as the sphere on the left by pinching the ends together. Gluing along the slits gives the double cover: i.e. $\pi : L_\lambda \to \mathbb{C}P^1$ as a degree-2 ramified cover.



Pictures above are from [12].

For given λ, it has a unique up to scalar holomorphic differential 1-form

$$\omega_\lambda := \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}}.$$

With ω_λ, one can compute periods of E_λ as follows. Its first homology group $H_1(L_\lambda,\mathbb{Z})$ is a rank-2 $\mathbb{Z}$-module with two generators, say $H_1(L_\lambda,\mathbb{Z}) = \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$, one of the generators, say γ_1 can be chosen as the blue circle γ_{01}, which is homotopic to $\gamma_{1/\lambda\infty}$. The other generator can be chosen as $\gamma_{0\infty}$. Namely a set of generators for $H_1(L_\lambda,\mathbb{Z})$ can be assemble from $\pi^{-1}(\gamma_{P_iP_j})$ where the $\gamma_{P_iP_j}$ are paths on $\mathbb{C}P^1$ connecting branched points P_i, P_j where $1 \le i < j \le 4$.

The *periods* of L_λ is the lattice

$$\Lambda(\lambda) = \mathbb{Z}\int_{\gamma_1} \omega_\lambda \oplus \mathbb{Z}\int_{\gamma_2} \omega_\lambda.$$

If a different basis $\{\gamma_1',\gamma_2'\}$ of $H_1(L_\lambda,\mathbb{Z})$ is chosen, then $\begin{pmatrix}\gamma_1'\\ \gamma_2'\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$,

$$\tau' = \int_{\gamma_2'} \omega_\lambda \Big/ \int_{\gamma_1'} \omega_\lambda = \begin{pmatrix} d & c\\ b & a\end{pmatrix} \cdot \tau = \frac{d\tau + c}{b\tau + a},$$

where $\begin{pmatrix} d & c \\ b & a \end{pmatrix} \in \Gamma(2)$. Upon relabelling, we can assume

$$\tau = \frac{\int_{\gamma_2} \omega_\lambda}{\int_{\gamma_1} \omega_\lambda} \in \mathbb{H}$$

and its isomorphism class is determined by the value of the modular $\lambda(\tau)$ evaluated at τ, which is invariant under linear transformation by elements in $\Gamma(2)$.

Question 1. *How likely is the period ratio $\tau \in \overline{\mathbb{Q}}$?*

The answer is not very likely, unless L_λ admits complex multiplication (CM), which is a fundamental result by Schneider, see [24].

Theorem 1.21 (Chowla and Selberg)**.** *If E is an elliptic curve whose endomorphism ring over $\mathbb{C}$ is an order of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ with fundamental discriminant $-d$, then all periods of E are algebraic multiples of a particular transcendental number*

$$\omega_{-d} := \Gamma\left(\frac{1}{2}\right) \prod_{0<a<d} \Gamma\left(\frac{a}{d}\right)^{\frac{n\epsilon(a)}{4h_K}}, \tag{24}$$

where $\Gamma(\cdot)$ stands for the Gamma function, n is the number of torsion elements in K, ϵ is the primitive quadratic Dirichlet character modulo d, that is, the quadratic character attached to K over $\mathbb{Q}$, and h_K is the class number of K

See [26] by Selberg and Chowla, [14] by Gross, or [37, (97)].

Exercise 1.3. Verify that $\omega_{-4} = \Gamma\left(\frac{1}{2}\right) \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$ and $\omega_{-3} = \Gamma\left(\frac{1}{2}\right) \left(\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}\right)^{3/2}$.

One take-way from this theorem is that ratios alike might come from CM background.

Exercise 1.4. Compute ω_{-d} for $d = 7, 8, 11, 19, 43, 67, 163$.

We now return to the discussion of periods.

Question 2. *How is the above picture related to the differential equation $L_{\{1/2,1/2\},\{1,1\}\lambda} F = 0$?*

For any $\gamma \in H_1(L_\lambda, \mathbb{Z})$, define $p_\gamma(\lambda) := \displaystyle\int_\gamma \omega_\lambda$. If we vary γ analytically, $p_\gamma(\lambda)$ becomes a function of λ. This means by analytic continuation, $\tau(\lambda) = \displaystyle\int_{\gamma_2} \omega_\lambda / \int_{\gamma_1} \omega_\lambda$ is also considered as function of λ. The differential equation $L_{\{1/2,1/2\},\{1,1\}\lambda} F = 0$ is called the *Picard-Fuchs equation* of L_λ, by which it means $L_{\{1/2,1/2\};\{1,1\}} \omega_\lambda$ is an exact form on L_λ.

Exercise 1.5. Verify that $L_{\{1/2,1/2\};\{1,1\}} \omega_\lambda$ is an exact form on L_λ.

It follows that both $\displaystyle\int_{\gamma_1} \omega_\lambda$ and $\displaystyle\int_{\gamma_2} \omega_\lambda$ are solutions of $L_{\{1/2,1/2\},\{1,1\};\lambda} F = 0$, which are linearly independent as functions of λ.

For example if we compute $\int_{\gamma_{01}} \omega_\lambda$, up to a scalar, by Euler integral formula, it agrees with $2\int_0^1 \omega_\lambda = 2 \cdot {}_2P_1 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{bmatrix}; \lambda \end{bmatrix} = 2\pi \cdot {}_2F_1 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{bmatrix}; \lambda \end{bmatrix}$. The other linearly independent solution is ${}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; 1-\lambda\right]$ (see Exercise 1.2). This explains the claim in Theorem 1.20 that the Schwarz map is the inverse of λ-function.

In the literature, there is another model of the Legendre curve which is more convenient for elliptic integrals. Given $k \in \mathbb{C} \smallsetminus \{0, \pm 1\}$, let

$$L(k): \quad y^2 = (1-x^2)(1-k^2x^2)$$

which is now a double cover of $\mathbb{C}P^1$ ramified at $\pm 1, \pm 1/k$. Similar to the above, one can perform branch cuts along two non-intersecting lines between 1 and -1 as well as between $1/k$ and $-1/k$. The 1st de Rham cohomology $H^1_{DR}(L(k)/\mathbb{C})$ is isomorphic to the differentials on $L(k)$ with at most a double pole at infinity (see [15, Appendix 1] by Katz). It is a 2-dimensional vector space over $\mathbb{C}$ generated by $\mathfrak{w}_1(k) = \frac{dx}{y}$ and $\mathfrak{w}_2(k) = (1-k^2x^2)\frac{dx}{y}$. Here $\mathfrak{w}_1(k)$ is holomorphic. Typically there is no a priori 'functorial' choice for the second generator of $H^1_{DR}(L(k)/\mathbb{C})$ unless $L(k)$ admits complex multiplication (CM), see [15]. Like the above model, a loop γ on $L(k)$ can be obtained from the path from -1 to 1 wrapping around twice. The two periods $\int_L \mathfrak{w}_i(k)$, $i = 1, 2$ can be computed by

$$K(k) = \int_0^1 \mathfrak{w}_1(k) = \frac{\pi}{2}\, {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix}\, ;\, k^2\right]$$

and

$$E(k) = \int_0^1 \mathfrak{w}_2(k) = \frac{\pi}{2}\, {}_2F_1\left[\begin{matrix} -\frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix}\, ;\, k^2\right],$$

which are *complete elliptic integrals of first and second kind* respectively, see [4, Chapter 1] by Borwein-Borwein.

Exercise 1.6. Verify that

$$E(k) = k(1-k^2)\frac{d}{dk}K(k) + (1-k^2)K(k).$$

They satisfy the Legendre relation (Theorem 1.6 [4])

$$E(k)K(k') + E(k')K(k) - K(k)K(k') = \frac{\pi}{2}, \quad k' = \sqrt{1-k^2}. \tag{25}$$

Remark 4. *The Legendre relation is an expression for the Wronskian of the differential equation* $\mathcal{L}_{\{\frac{1}{2},\frac{1}{2}\},\{1,1\};k^2}$.

When k is a singular moduli value, namely $L(k)$ admits CM, using the Clausen formula (see (28) below) and based on the Chowla and Selberg theorem 1.21, Borwein-Borwein gave a method to prove Ramanujan formulas for $1/\pi$ in [4]. Similar method was used by Chudnovsky-Chudnovsky [5], see [38] by Zudilin or [12, §7.2] for reviews of Ramanujan's formulas for $1/\pi$. One of Ramanujan's $1/\pi$ formulas is

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3}(6k+1)\frac{1}{4^k} = \frac{4}{\pi}. \tag{26}$$

1.10. **Hypergeometric formulas.** To obtain hypergeometric formulas, there are a variety of ways. Here we only summarize only a few that are relevant to our later discussion. Note that we omit the convergence conditions here. Careful readers can check them out from [1].

1.10.1. *From rigidity.* For instance, as a consequence of (21) plus the uniqueness of the local holomorphic solution near 0 with exponent 0, one has the following identify due to Pfaff

$${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\, ;\, x\right] = (1-x)^{-a}\, {}_2F_1\left[\begin{matrix} a & c-b \\ & c \end{matrix}\, ;\, \frac{x}{x-1}\right]. \tag{27}$$

Another example is the Clausen formula

$$(28)\qquad {}_2F_1\left[\begin{matrix} a & b \\ & a+b+\frac{1}{2} \end{matrix}; z\right]^2 = {}_3F_2\left[\begin{matrix} 2a & 2b & a+b \\ & 2a+2b & a+b+\frac{1}{2} \end{matrix}; z\right].$$

Note again both hand sides take value 1 at $z=0$. The scheme for the ${}_2F_1$ on the left is

$$P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & a & ; & z \\ \frac{1}{2}-a-b & \frac{1}{2} & b & & \end{pmatrix}$$

Its symmetric square is

$$P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & 2a & ; & z \\ \frac{1}{2}-a-b & 1 & 2b & & \\ 1-2a-2b & \frac{1}{2} & a+b & & \end{pmatrix},$$

which coincides with the Riemann scheme of the right hand side.

This technique is also known as "pull-back transformation between hypergeometric differential equations", see [31] by Vidunas. A general pull-back transformation of higher degree converts a hypergeometric differential equation to a Fuchsian equation with several singularities. A pull-back of second order hypergeometric equations to a Fuchsian equation with three singularities gives rise to an algebraic transformation of ${}_2F_1$-functions.

Example 1.8. The identities of the local systems

$$P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & \frac{a}{2} & ; & z^2 \\ \frac{1}{2}-b & b-a & \frac{a+1}{2} & & \end{pmatrix} = P\begin{pmatrix} 0 & -1 & 1 & \infty & & \\ 0 & 0 & 0 & a & ; & z \\ 1-2b & b-a & b-a & a+1 & & \end{pmatrix}$$
$$= P\begin{pmatrix} 0 & 1 & \infty & 2 & & \\ 0 & 0 & 0 & a & ; & \frac{2z}{z+1} \\ 1-2b & b-a & b-a & a+1 & & \end{pmatrix} = \left(2-\frac{2z}{z+1}\right)^a P\begin{pmatrix} 0 & 1 & \infty & & \\ 0 & 0 & a & ; & \frac{2z}{z+1} \\ 1-2b & b-a & b & & \end{pmatrix}$$

give one of the Kummer quadratic formulas:

$$(1+z)^a\,{}_2F_1\left[\begin{matrix} \frac{a}{2} & \frac{a+1}{2} \\ & \frac{1}{2}+b \end{matrix}; z^2\right] = {}_2F_1\left[\begin{matrix} a & b \\ & 2b \end{matrix}; \frac{2z}{z+1}\right]$$

equivalently,

$${}_2F_1\left[\begin{matrix} a & b \\ & 2b \end{matrix}; z\right] = \left(1-\frac{z}{2}\right)^{-a} {}_2F_1\left[\begin{matrix} \frac{a}{2} & \frac{a+1}{2} \\ & \frac{1}{2}+b \end{matrix}; \left(\frac{z}{2-z}\right)^2\right].$$

To double check we see that as formal power series near $z=0$, both hand sides start with constant 1.

Similarly, we have

$$P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & ; \ z^2 \\ \frac12 & \frac12-a-b & b & \end{pmatrix} = P\begin{pmatrix} -1 & 1 & \infty & \\ 0 & 0 & 2a & ; \ z \\ \frac12-a-b & \frac12-a-b & 2b & \end{pmatrix}$$
$$= P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & 2a & ; \ \frac{1+z}{2} \\ \frac12-a-b & \frac12-a-b & 2b & \end{pmatrix}$$
$$= P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & 2a & ; \ \frac{1-z}{2} \\ \frac12-a-b & \frac12-a-b & 2b & \end{pmatrix},$$

and hence there are constants C_1, C_2 such that

$${}_2F_1\begin{bmatrix} a & b & \\ & \frac12 & ; \ z \end{bmatrix} = C_1 \cdot {}_2F_1\begin{bmatrix} 2a & 2b & \\ & a+b+\frac12 & ; \ \dfrac{1-\sqrt{z}}{2} \end{bmatrix} + C_2 \cdot {}_2F_1\begin{bmatrix} 2a & 2b & \\ & a+b+\frac12 & ; \ \dfrac{1+\sqrt{z}}{2} \end{bmatrix}.$$

Exercise 1.7. 1) Can ${}_2F_1\begin{bmatrix} 2a & 2b & \\ & a+b+\frac12 & ; \ \frac{1-\sqrt{z}}{2} \end{bmatrix}$ and ${}_2F_1\begin{bmatrix} 2a & 2b & \\ & a+b+\frac12 & ; \ \frac{1+\sqrt{z}}{2} \end{bmatrix}$ be scalar multiple of each other?

2) What information would be helpful in order to reach the following conclusion

$$2\Gamma\left(\frac{\frac12, a+b+\frac12}{a+\frac12, b+\frac12}\right) {}_2F_1\begin{bmatrix} a & b & \\ & \frac12 & ; \ z \end{bmatrix} = {}_2F_1\begin{bmatrix} 2a & 2b & \\ & a+b+\frac12 & ; \ \dfrac{1-\sqrt{z}}{2} \end{bmatrix} + {}_2F_1\begin{bmatrix} 2a & 2b & \\ & a+b+\frac12 & ; \ \dfrac{1+\sqrt{z}}{2} \end{bmatrix},$$

where $\Gamma\left(\frac{a_1,\cdots,a_r}{b_1,\cdots,b_s}\right) = \frac{\Gamma(a_1)\cdots\Gamma(a_r)}{\Gamma(b_1)\cdots\Gamma(b_s)}$?

See the works of Goursat, Vidūnas [13, 31, 32, etc.] for more examples and details.

1.10.2. *From definition.* Euler integral formula immediately implies the Gauss evaluation formula

$$(29)\qquad {}_2F_1\begin{bmatrix} a & b & \\ & c & ; \ 1 \end{bmatrix} = \frac{1}{B(b,c-b)}\int_0^1 (1-t)^{-a} t^{b-1}(1-t)^{c-b-1}dt = \frac{B(b,c-a-b)}{B(b,c-b)} = \Gamma\left(\frac{c, c-a-b}{c-a, c-b}\right).$$

Another way to prove the Pfaff formula (27) is to use Euler's integral formula and then change variable which we recall here (following the proof of Theorem 2.2.5 of [1]).

$$\begin{aligned} {}_2F_1\begin{bmatrix} a & b & \\ & c & ; \ x \end{bmatrix} &= \frac{1}{B(b,c-b)}\int_0^1 (1-xt)^{-a} t^{b-1}(1-t)^{c-b-1}dt \\ &\overset{t\mapsto 1-s}{=} \frac{1}{B(b,c-b)}\int_0^1 (1-x+xs)^{-a}(1-s)^{b-1}s^{c-b-1}ds \\ &= \frac{(1-x)^{-a}}{B(b,c-b)}\int_0^1 (1-\frac{xs}{x-1})^{-a}s^{c-b-1}(1-s)^{b-1}ds \\ &= (1-x)^{-a}\,{}_2F_1\begin{bmatrix} a & c-b & \\ & c & ; \ \dfrac{x}{x-1} \end{bmatrix}. \end{aligned}$$

The Euler formula below is a consequence of the Pfaff formula iterated twice.

$$(30)\qquad {}_2F_1\begin{bmatrix} a & b & \\ & c & ; \ x \end{bmatrix} = (1-x)^{c-a-b}\,{}_2F_1\begin{bmatrix} c-a & c-b & \\ & c & ; \ x \end{bmatrix}.$$

Exercise 1.8. Derive (30) from (27).

1.10.3. *From comparing coefficients.* Obtaining evaluating formulas from comparing coefficients of known identities is commonly used. For example the Pfaff-Saalschütz formula [1, Thm. 2.2.6] follows from comparing coefficients on both sides of (30). It states that for a positive integer n and $a, b, c \in \mathbb{C}$,

(31)
$$ {}_3F_2\left[\begin{matrix} a & b & -n \\ & c & 1+a+b-n-c \end{matrix}; 1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} = \Gamma\left(\frac{c-a+n, c-b+n, c, c-a-b}{c-a, c-b, c+n, c-a-b+n}\right). $$

By Definition 6, the left hand side is a balanced series. In some sense, transformation formula like (30) and evaluation formula (31) are two equivalent ways to convey the same information.

Exercise 1.9. Assume (31), show that (30) holds as an equality between two formal power series.

In Euler transformation (30), if we use variable d, e, f in place of a, b, c, namely

$$ (1-x)^{f-d-e}\, {}_2F_1\left[\begin{matrix} f-d & f-e \\ & f \end{matrix}; x\right] = {}_2F_1\left[\begin{matrix} d & e \\ & f \end{matrix}; x\right] $$

and assume $c-a-b = f-d-e$, then

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; x\right] {}_2F_1\left[\begin{matrix} f-d & f-e \\ & f \end{matrix}; x\right] = {}_2F_1\left[\begin{matrix} c-a & c-b \\ & c \end{matrix}; x\right] {}_2F_1\left[\begin{matrix} d & e \\ & f \end{matrix}; x\right] $$

From comparing coefficients and relabelling the variables, one gets the following formula relating two balanced ${}_4F_3$ series

(32) $$ {}_4F_3\left[\begin{matrix} -n & a & b & c \\ & d & e & f \end{matrix}; 1\right] = \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}\, {}_4F_3\left[\begin{matrix} -n & a & d-b & d-c \\ & d & a+1-n-e & a+1-n-f \end{matrix}; 1\right], $$

where $a+b+c-n+1 = d+e+f$. See Theorem 3.3.3 [1].

Let $f \to \infty$ while keeping $f-c$ fixed. This means $\frac{(c)_k}{(f)_k} \to 1$ for any $k \le n$, which is a fixed finite positive integer. Under this assumption, ${}_4F_3\left[\begin{matrix} -n & a & b & c \\ & d & e & f \end{matrix}; 1\right] \mapsto {}_3F_2\left[\begin{matrix} -n & a & b \\ & d & e \end{matrix}; 1\right]$. Dealing with the right hand side similarly leads to

(33) $$ {}_3F_2\left[\begin{matrix} -n & a & b \\ & d & e \end{matrix}; 1\right] = \frac{(e-a)_n}{(e)_n}\, {}_3F_2\left[\begin{matrix} -n & a & d-b \\ & d & a+1-n-e \end{matrix}; 1\right]. $$

Long, Osburn and Swisher used (33) and the Pfaff-Saalschütz formula (31) to prove a conjecture of Kimoto and Wakayam, see [19] for more information.

Exercise 1.10. This is related to (32). Check that

$$ \iota : (-n, a, b, c\,;\, d, e, f) \mapsto (-n, a, d-a, d-c\,;\, d, a+1-n-e, a+1-n, f) $$

is an involution on the 7-tuples of parameters in which the upper parameters and lower parameters are separated by ";". In addition, one can permute the upper (resp. lower) parameters. Say you permute a, b on the left and then apply ι again, what will you get? Do you get a new ${}_4F_3(1)$ series?

See [33] by Whipple for relations among ${}_3F_2(1)$ series and [11] by Formichella, Green and Stade for Coxeter group actions on balanced ${}_4F_3(1)$ series.

1.10.4. *From local expansion agreements.* The idea behind the method is very straightforward, namely two formal power series are identical if their coefficients are the same.

We will illustrate how to use it to prove the following Kummer quadratic transformation formula

$$(1-x)^{-c}\ {}_2F_1\left[\begin{matrix}\frac{1+c}{2}-b & \frac{c}{2} \\ & c-b+1\end{matrix};\ \frac{-4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix}b & c \\ & c-b+1\end{matrix};\ x\right]. \tag{34}$$

We now outline a proof using following [12] by noting

$$(a)_{n-r} = (-1)^r \frac{(a)_n}{(1-a-n)_r}. \tag{35}$$

Proof. To begin, note that

$$\begin{aligned}\binom{-c-2k}{n-k} &= (-1)^{n-k}\frac{(c+2k)_{n-k}}{(1)_{n-k}} \overset{(35)}{=} (-1)^{n-k}\frac{(c+2k)_n(-n)_k}{(1-c-2k-n)_k(1)_n} \\ &= (-1)^{n-k}\frac{\Gamma(c+2k+n)\Gamma(1-c-2k-n)}{\Gamma(c+2k)\Gamma(1-c-k-n)}\frac{(-n)_k}{n!} \\ &\overset{\text{reflection}}{=} (-1)^n\frac{\Gamma(c+2k+n)\Gamma(c+k+n)}{\Gamma(c+2k)\Gamma(c+2k+n)}\frac{(-n)_k}{n!} = (-1)^n\frac{(c)_{n+k}(-n)_k}{(c)_{2k}n!} \\ &= (-1)^n\frac{(c)_n(c+n)_k(-n)_k}{(c)_{2k}n!} \overset{(6)}{=} (-1)^n\frac{(c)_n(c+n)_k(-n)_k}{4^k(\frac{c}{2})_k(\frac{c+1}{2})_k n!}.\end{aligned} \tag{36}$$

The left hand side of (34) can be expanded as

$$\begin{aligned} & \sum_{k\ge 0}\frac{(\frac{1+c}{2}-b)_k(\frac{c}{2})_k}{k!(c-b+1)_k}(-4x)^k(1-x)^{-c-2k} \\ = & \sum_{k\ge 0}\frac{(\frac{1+c}{2}-b)_k(\frac{c}{2})_k}{k!(c-b+1)_k}(-4x)^k\sum_{i\ge 0}\binom{-c-2k}{i}(-x)^i \\ \overset{n=k+i}{=} & \sum_{k,n\ge 0}\frac{(\frac{1+c}{2}-b)_k(\frac{c}{2})_k}{k!(c-b+1)_k}4^k\binom{-c-2k}{n-k}(-x)^n \\ \overset{(36)}{=} & \sum_{k,n\ge 0}\frac{(\frac{1+c}{2}-b)_k(\frac{c}{2})_k}{k!(c-b+1)_k}4^k\frac{(c)_n(c+n)_k(-n)_k}{4^k(\frac{c}{2})_k(\frac{c+1}{2})_k n!}x^n \\ = & \sum_{n\ge 0}\frac{(c)_n}{n!}\ {}_3F_2\left[\begin{matrix}\frac{1+c}{2}-b & c+n & -n \\ & \frac{c+1}{2} & c-b+1\end{matrix};\ 1\right]x^n.\end{aligned}$$

By the Pfaff-Saalschütz formula (31), the above equals

$$\sum_{n\ge 0}\frac{(c)_n(b)_n(\frac{1-c}{2}-n)_n}{n!(\frac{1+c}{2})_n(b-c-n)_n}x^n \overset{\text{reflection}}{=} \sum_{n\ge 0}\frac{(b)_n(c)_n}{n!(c-b+1)_n}x^n.$$

□

Exercise 1.11. Prove the following Bailey cubic transformation formula:

$${}_2F_1\left[\begin{matrix}a & \frac{1-a}{3} \\ & \frac{4a+5}{6}\end{matrix};\ x\right] = (1-4x)^{-a}\ {}_2F_1\left[\begin{matrix}\frac{a}{3} & \frac{a+1}{3} \\ & \frac{4a+5}{6}\end{matrix};\ \frac{-27x}{(1-4x)^3}\right]. \tag{37}$$

1.10.5. *From specializing values in known identities.* For example, if we let $x = -1$ in (34), then the left hand side becomes $2^{-c}\,{}_2F_1\left[\begin{matrix}\frac{1+c}{2}-b & \frac{c}{2} \\ & c-b+1\end{matrix}\,;\,1\right]$ which can be evaluated by the Gauss evaluation formula (29). As a consequence, it leads to the Kummer's evaluation formula (See Corollary 3.1.2 of [1])

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}b & c \\ & c-b+1\end{matrix}\,;\,-1\right] &= 2^{-c}\,{}_2F_1\left[\begin{matrix}\frac{1+c}{2}-b & \frac{c}{2} \\ & c-b+1\end{matrix}\,;\,1\right] \\
&= 2^{-c}\Gamma\left(\frac{c-b+1,\frac{1}{2}}{\frac{c+1}{2},1+\frac{c}{2}-b}\right) = \Gamma\left(\frac{1+c-b,\frac{c}{2}+1}{1+c,\frac{c}{2}-b+1}\right).
\end{aligned}
\tag{38}
$$

Exercise 1.12. Work out the last equality.

By Definition (5), the left hand side is a well-posed series evaluated at -1.

1.10.6. *From Bailey transform.* There are other nice methods described in the textbooks including Bailey transform [27, 2.4], we will leave the interested readers to check the details. But we would like to mention as a consequence, one can prove the following formula originally discovered by Whipple in [34]. It says

$$
{}_7F_6\left[\begin{matrix}a & 1+\frac{a}{2} & c & d & e & f & g \\ & \frac{a}{2} & 1+a-c & 1+a-d & 1+a-e & 1+a-f & 1+a-g\end{matrix}\,;\,1\right]
\tag{39}
$$

$$
= \Gamma\left(\frac{1+a-e,1+a-f,1+a-g,1+a-e-f-g}{1+a,1+a-f-g,1+a-e-f,1+a-e-g}\right)\cdot {}_4F_3\left[\begin{matrix}a & e & f & g \\ & e+f+g-a & 1+a-c & 1+a-d\end{matrix}\,;\,1\right],
$$

when both sides terminate. Again the left hand side is a well-posed series while the ${}_4F_3(1)$ series on the right hand side is 2-balanced.

1.10.7. *From formulas to identities.* As the audience may already notice, these hypergeometric formulas give a rich source of identities. For instance, if we let $a, b = -n$ in Gauss evaluation 29, we will get immediately

$$
\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k} = \binom{n+m}{n}.
\tag{40}
$$

Exercise 1.13. Guess a formula for $\displaystyle\sum_{k=0}^{n}\binom{n}{k}^2(-1)^k$ and then prove your claim.

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