Hypergeometric Functions, Character Sums and Applications, Part I

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University of Connecticut 2021 Summer Lecture Series

https://alozano.clas.uconn.edu/hypergeometric

June 29, 2021

Plan

Day 1. Hypergeometric functions over $\mathbb{C}$

- 1.I Hypergeometric functions and differential equations
- 1.II Hypergeometric formulas and Legendre curves

Day 2. Over finite fields

- 2.I Hypergeometric functions over finite fields
- 2.II Point counts over finite fields

Day 3. In Galois perspective

- 3.I Hypergeometric Galois representations
- 3.II Modularity results

Day 4. p-adic hypergeometric functions and supercongruences

- 4.I Dwork unit roots
- 4.II Supercongruences

Notation

Pochhammer symbol

$$(a)_n := a(a+1)\cdots(a+n-1), \quad (a)_0 = 1. \tag{1}$$

$$\frac{(-1)^k(-n)_k}{k!} = \binom{n}{k}. \tag{2}$$

For $\mathrm{Re}(x) > 0$,

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t.$$

It can be extended to a meromorphic function, satisfying

$$\Gamma(x+1) = x\Gamma(x), \tag{3}$$

For $n \in \mathbb{Z}_{\geq 0}$,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{4}$$

Key properties of the Gamma function

Reflection formula

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}, \quad \forall a \in \mathbb{C}$$

Multiplication formula

$$\Gamma(2a)(2\pi)^{1/2} = 2^{2a-\frac{1}{2}}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right), \quad \forall a \in \mathbb{C}. \tag{5}$$

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n. \tag{6}$$

Theorem (Nesterenko)

For any imaginary quadratic field with discriminant $-d$ and character $\epsilon(\cdot) = \left(\frac{-d}{\cdot}\right)$, the numbers

$$\pi, \quad e^{\pi\sqrt{d}}, \quad \prod_{0<a<d}\Gamma\left(\frac{a}{d}\right)^{\epsilon(a)}$$

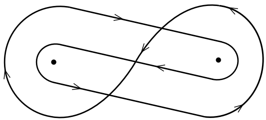
are algebraically independent.

Beta function

For $\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0$

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

The assumptions on x and y can be relaxed by integrating along the Pochhammer contour path around 0 and 1.



Integrating over the double contour loop γ_{01}, the integral

$$B(x, y) = \frac{1}{(1-e^{2\pi i x})\,(1-e^{2\pi i y})} \int_{\gamma_{01}} t^{x-1}(1-t)^{y-1}\mathrm{d}t$$

Hypergeometric parameters

A multi-set $\alpha = \{a_1, ..., a_n\}$ with $a_i \in \mathbb{Q}$, elements can repeat.
It is called *defined* over $\mathbb{Q}$, if $\prod_{j=1}^{n}(X - e^{2\pi i a_j}) \in \mathbb{Z}[X]$. It is said to be *self-dual* if $\alpha \equiv -\alpha \mod \mathbb{Z}$.
E.g.

A set of hypergeometric parameters consists of

$$\alpha = \{a_1, ..., a_n\}, \beta = \{b_1 = 1, b_2, ..., b_n\}$$

with $a_i, b_j \in \mathbb{Q}$. It is called *primitive* if $a_i - b_j \notin \mathbb{Z}$ for any i, j.
Let $M := lcd(\alpha, \beta)$ the least positive common denominators of a_i, b_j's.

Hypergeometric functions

Given $\alpha = \{a_1, ..., a_n\}, \beta = \{b_1 = 1, b_2, ..., b_n\}$

$$F(\alpha, \beta; z) = {}_nF_{n-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_n \\ & b_2 & \cdots & b_n \end{matrix}; \ z\right] := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_n)_k}{(b_1)_k \cdots (b_n)_k} z^k. \tag{7}$$

Period functions

Let

$$ {}_1P_0\left[a_1\ ;\ z\right] := (1-z)^{-a_1} $$

Inductively

$$ {}_{n+1}P_n\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_{n+1} \\ & b_2 & \cdots & b_{n+1} \end{array} ; z\right] := $$
$$ \int_0^1 t^{a_{n+1}-1}(1-t)^{b_{n+1}-a_{n+1}-1}\, {}_nP_{n-1}\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ & b_2 & \cdots & b_n \end{array} ; zt\right] \mathrm{d}t. \tag{8} $$

The order of the $a_i's$ (resp. $b_j's$) matters.
So

$$ {}_2P_1\left[\begin{array}{cc} a_1 & a_2 \\ & b_2 \end{array} ;\ z\right] = \int_0^1 x^{a_2-1}(1-x)^{b_2-a_2-1}(1-zx)^{-a_1}\,dx. $$

Euler integral formula

$$(1-zx)^{-a} = \sum_{k\geq 0} \binom{-a}{k} (zx)^k = \sum_{k\geq 0} \frac{(a)_k}{k!}(zx)^k.$$

$$\begin{aligned}
{}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix} ;\ z\right] &= \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx \\
&= \int_0^1 x^{b-1}(1-x)^{c-b-1}\sum_{k=0}^{\infty}\frac{(a)_k}{k!}(zx)^k dx \\
&= \sum_{k=0}^{\infty}\frac{(a)_k z^k}{k!}\int_0^1 x^{b-1+k}(1-x)^{c-b-1}dx \\
&= \sum_{k=0}^{\infty}\frac{(a)_k z^k}{k!}B(b+k, c-b) \\
&= B(b, c-b)\,{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix} ;\ z\right].
\end{aligned}$$

Normalized Period functions

Let

$$ {}_1F_0\left[a_1\ ;\ z\right] = {}_1P_0\left[a_1\ ;\ z\right] = (1-z)^{-a_1}. $$

$$ {}_{n+1}F_n\left[\begin{matrix} a_1 & a_2 & \cdots & a_{n+1} \\ & b_2 & \cdots & b_{n+1} \end{matrix}\ ;\ z\right] $$
$$ = \prod_{i=2}^{n+1} B(a_i, b_i - a_i)^{-1} \cdot {}_{n+1}P_n\left[\begin{matrix} a_1 & a_2 & \cdots & a_{n+1} \\ & b_2 & \cdots & b_{n+1} \end{matrix}\ ; z\right]. $$

1) The leading coefficient is 1;
2) The roles of the upper entries a_i (resp. lower entries b_j) are symmetric.

Hypergeometric differential equations

Let $\theta_z := z\frac{d}{dz}$.

Lemma

Let $F(\alpha,\beta;z) = \sum_{k\geq 0} A(k)z^k$ as before. Let

$$\mathcal{L}_{\alpha,\beta;z} := \prod_{i=1}^{n}(\theta_z + b_i - 1) - z\prod_{i=1}^{n}(\theta_z + a_i), \tag{9}$$

then

$$\mathcal{L}_{\alpha,\beta;z}(F(\alpha,\beta;z)) = 0.$$

$\mathcal{L}_{\alpha,\beta;\lambda}$ has only 3 regular singularities at 0, 1, ∞.

Example

$$\mathcal{L}_{\{a_1,a_2\},\{1,b_2\};\lambda} = D^2 + \frac{b_2 - (a_1 + a_2 + 1)\lambda}{\lambda(1-\lambda)} D - \frac{a_1 a_2}{\lambda(1-\lambda)}. \quad (10)$$

At singularity a, the *characteristic/indicial equation* is

$$r(r-1)\cdots(r-n+1) + C_{n-1} r(r-1)\cdots(r-n+2) + \cdots + C_0 = 0, \quad (11)$$

where $C_i = a_i(x)(x-a)^{n-i}|_{x=a}$. Roots of this polynomial are *characteristic exponents* at a.

E. g.

Rigidity

Theorem

The local exponents of $\mathcal{L}_{\alpha,\beta;\lambda}$ are

$$\begin{aligned} 0, 1-b_2, \cdots, 1-b_n \quad & \textit{at } \lambda = 0 \\ a_1,\ a_2,\ \cdots, a_n \quad & \textit{at } \lambda = \infty \\ 0, 1, 2, \cdots, n-2, \gamma \quad & \textit{at } \lambda = 1, \end{aligned} \tag{12}$$

where $\gamma = -1 + \sum_{j=1}^{n} b_j - \sum_{j=1}^{n} a_j$.

Remark

The local monodromy matrix M_1 is called a (quasi)reflection as the rank of $M_1 - I_n$ is 1.

Theorem (Rigidity Theorem)

Each order-n ordinary differential equation in variable z which has only three regular singularities at 0, 1, ∞ and the corresponding indicial exponents as (12) is equivalent to $\mathcal{L}_{\alpha,\beta;z}F = 0$.

Local solutions

The solution space of $\mathcal{L}_{\alpha,\beta;z}$ is n-dimensional.

Example

For $\mathcal{L}_{\alpha,\beta;z}$, around the singularity $z = 0$, if $b_2 \notin \mathbb{Z}$, a basis of the solution space can be given by

$$f_0 = {}_2F_1 \left[\begin{matrix} a_1 & a_2 \\ & b_2 \end{matrix} ; \ z \right]$$

$$g_0 = (z)^{1-b_2} {}_2F_1 \left[\begin{matrix} 1 + a_1 - b_2 & 1 + a_2 - b_2 \\ & 2 - b_2 \end{matrix} ; \ z \right].$$

When $b_2 \in \mathbb{Z}$, say $b_2 = 1$

$$g_0 = \log z \cdot f_0 + \sum_{m=1}^{\infty} \frac{(a_1)_m (a_2)_m}{m!^2} z^m \left[\sum_{k=1}^{m} \left(\frac{1}{a_1 + k - 1} + \frac{1}{a_2 + k - 1} - \frac{2}{k} \right) \right].$$

Monodromy representation

Given an order n ordinary Fuchsian differential equation L with only regular singularities $x_1, \cdots, x_s$, let $\pi_1(\mathbb{C}P^1 \setminus \{x_1, \cdots, x_s\}, x_0)$ denote its the fundamental group. By Cauchy, near x_0, the solution space of the homogeneous equation $Lu = 0$ is an n-dimensional vector space $V(x_0) = \langle f_1, \cdots, f_n \rangle$. Let $L(t) : [0, 1] \to \mathbb{C}P^1 \setminus \{x_1, \cdots, x_s\}$ be a continuous function with $L(0) = L(1) = x_0$. It is image is topologically a closed loop. We extend $f_1, \cdots, f_n$ analytically along L when t varies from 0 to 1. By Frobenius' result,

$$(f_1(L(1)), \cdots, f_n(L(1)))^T = M(L)(f_1, \cdots, f_n)^T$$

where $M(L) \in GL_n(\mathbb{C})$ only depending on the class $[L]$ of L in $\pi_1(\mathbb{C}P^1 \setminus \{x_1, \cdots, x_s\}, x_0)$. The map from

$$[L] \mapsto M(L)$$

is a homomorphism, which is unique up to conjugation by $GL_n(\mathbb{C})$. It is called a *monodromy representation* of L.

Beukers-Heckman's Theorem

Theorem (Beukers-Heckman)

Given primitive α, β, there is an explicit way to determine the hypergeometric monodromy group H and its Zariski closure $\overline{H}$. Here are the possibilities:

1. *H is finite*
2. *else if self-dual and $c = e^{2\pi i\gamma} = 1$, then $\overline{H} = Sp_n(\mathbb{C})$;*
3. *else if self-dual and $c = e^{2\pi i\gamma} = -1$, then $\overline{H} = O_n(\mathbb{C})$;*
4. *otherwise $SL_n(\mathbb{C}) \subset \overline{H}$.*

Corollary

If α, β are defined over $\mathbb{Q}$ and the sets $\{e^{2\pi i a_j}\}_{1\leq j\leq n}$ and $\{e^{2\pi i b_j}\}_{1\leq j\leq n}$ interlace, then H is finite.

The hypergeometric group for the multi-sets
$\alpha = \{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}\}, \beta = \{1, \frac{1}{3}, \frac{2}{3}\}$ is finite;
while hypergeometric group for the multi-sets
$\alpha = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}, \beta = \{1, \frac{1}{6}, \frac{5}{6}\}$ is $O_3(\mathbb{C})$.

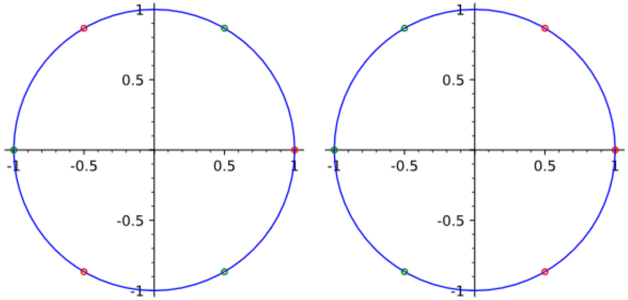


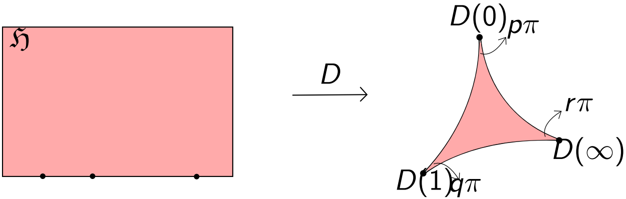
Figure 1: $\{\alpha = \{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}\}, \beta = \{1, \frac{1}{3}, \frac{2}{3}\}\}$, $\{\alpha = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}, \beta = \{1, \frac{1}{6}, \frac{5}{6}\}\}$

Schwarz theorem

Fix a $z_0 \in \mathbb{H}$, let f, g be two independent solutions to $\mathcal{L}_{\{a,b\},\{1,c\};z} F = 0$ near at z_0, and let $p = |1-c|$, $q = |c-a-b|$, and $r = |a-b|$. If p, q, $r < 1$, then the Schwarz map

$$D: \quad \mathbb{H} \cup \mathbb{R} \longrightarrow \mathbb{P}^1, \quad D(z) = f/g(z)$$

gives a bijection from $\mathbb{H} \cup \mathbb{R}$ onto a curvilinear triangle with vertices $D(0)$, $D(1)$, $D(\infty)$ and corresponding angles $p\pi$, $q\pi$, $r\pi$.



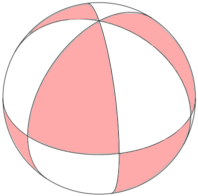
The universal cover S of the Schwarz triangle is

- sphere ($\mathbb{P}^1$) if $p+q+r > 1$ (finite monodromy);
- the Euclidean plane ($\mathbb{C}$) if $p+q+r = 1$;
- the hyperbolic plane ($\mathbb{H}$) if $p+q+r < 1$.

A spherical/algebraic case

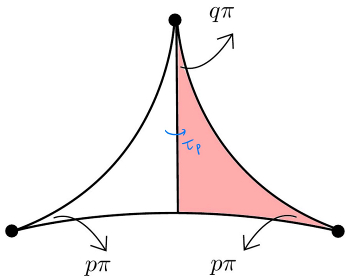
$$ {}_2F_1\left[\begin{matrix}\frac{1}{6} & \frac{2}{3} \\ & \frac{1}{2}\end{matrix}\,;\ z\right] = \frac{1}{2}\left((1+\sqrt{z})^{-\frac{1}{3}} + (1-\sqrt{z})^{-\frac{1}{3}}\right), $$

In which case $p = \frac{1}{2}, q = \frac{1}{3}, r = \frac{1}{2}$, the universal cover is



Triangle groups

We now assume that $|1-c| = p = 1/e_1$, $|c-a-b| = q = 1/e_2$, and $|a-b| = r = 1/e_3$ with $e_i = 2, \ldots, \infty$. Let τ_p, τ_q, τ_r be reflections in the edges of the Schwarz triangle $\Delta(p,q,r)$.



Set $g_p = \tau_q\tau_r$, $g_q = \tau_p\tau_r$, $g_r = \tau_p\tau_q$. Then the triangle group

$$\Gamma = (e_1, e_2, e_3) := \langle g_p, g_q, g_r \,:\, g_p^{e_1} = g_q^{e_2} = g_r^{e_3} = g_p g_q g_r = 1\rangle$$

is a subgroup of $\mathrm{Isom}^+(S)$.

Arithmetic triangle groups

When $p + q + r < 1$, we say that $\Gamma = (e_1, e_2, e_3)$ is *arithmetic* if Γ is commensurable with norm 1 group of certain quaternion order. There are exactly 85 arithmetic triangle groups falling into 19 classes by Takeuchi. When $X(\Gamma)$ is compact, it is called a Shimura curve, according to Shimura, it parametrizes certain abelian surfaces with quaternionic multiplication (QM).

Example

- $(2, 3, \infty) \simeq \mathrm{PSL}_2(\mathbb{Z})$
- $(\infty, \infty, \infty) \simeq \Gamma(2)$
- The monodromy group for $\alpha = \{\frac{1}{6}, \frac{1}{3}\}$ and $\beta = \{1, \frac{5}{6}\}$ is isomorphic to (3,6,6)

Question

Given an arithmetic triangle group Γ, how to find a model for the "universal" 2-dimensional abelian varieties parameterized by $\mathbb{H}/\Gamma$?

Theorem (Deines, Fuselier, Long, Swisher, Tu)

Let $N = 3, 4, 6$, i, j, k be integers between 1 to $N - 1$ such that $gcd(i, j, k) = 1$ and $N \nmid i, k, j, i + j + k$. Then for each $\lambda \in \overline{\mathbb{Q}}$, the endomorphism algebra of the primitive part of the 2-dimensional abelian variety constructed from the Jacobian of the smooth model of $y^N = x^i(1 - x)^j(1 - \lambda x)^k$ contains a quaternion algebra over $\mathbb{Q}$ if and only if the beta quotient

$$B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) \Big/ B\left(\frac{k}{N}, \frac{2N-i-j-k}{N}\right) \in \overline{\mathbb{Q}}.$$

Corollary

From $y^6 = x^4(1 - x)^3(1 - \lambda x)^1$, we get a 1-parameter family of 2-dim'l abelian varieties whose endomorphism algebra over $\mathbb{Q}$ contains $\left(\frac{-3,2}{\mathbb{Q}}\right)$. [A different construction by Petkova and Shiga.]

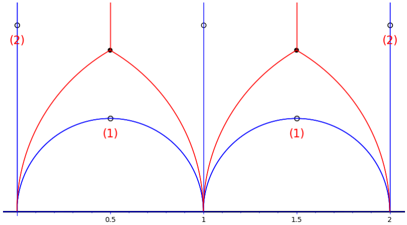
A Schwarz map example

The Schwarz map $D(\lambda) := i\,\dfrac{{}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; 1-\lambda\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda\right]}$

of $L_{\{1/2,1/2\};\{1,1\}}$ gives an isomorphism

$$\mathbb{P}^1 - \{0, 1, \infty\} \longrightarrow \Gamma(2)\backslash\mathbb{H},$$

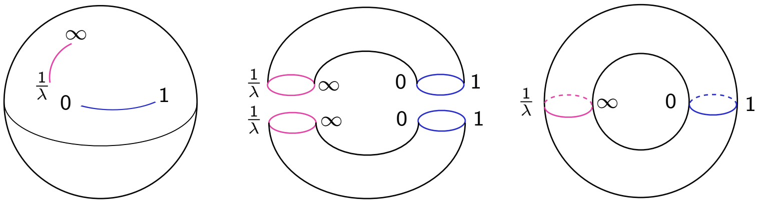
which is the inverse map of the modular λ-function.



The Legendre curves

$$L_\lambda : y^2 = x(1-x)(1-\lambda x). \tag{13}$$

When $\lambda \neq 0, 1$ it is an elliptic curve. It is a double over of $\mathbb{C}P^1$ which ramifies only at $P_1 = 0, P_2 = 1, P_3 = \frac{1}{\lambda}, P_4 = \infty$ as demonstrated by the picture below. Gluing along the slits gives the double cover: i.e. $\pi : L_\lambda \to \mathbb{C}P^1$ as a degree-2 ramfied cover.



For given λ, it has a unique up to scalar homolorphic differential 1-form

$$\omega_\lambda := \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}}.$$

Periods

$H_1(L_\lambda, \mathbb{Z}) = \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$, γ_{01} and $\gamma_{0\infty}$.
The *period* lattice

$$\Lambda(\lambda) = \mathbb{Z}\int_{\gamma_1} \omega_\lambda \oplus \mathbb{Z}\int_{\gamma_2} \omega_\lambda.$$

If $\{\gamma_1', \gamma_2'\}$ is a different basis, then $\begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$,

$$\tau' = \int_{\gamma_2'} \omega_\lambda / \int_{\gamma_1'} \omega_\lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \cdot \tau = \frac{d\tau + c}{b\tau + a},$$

where $\begin{pmatrix} d & c \\ b & a \end{pmatrix} \in \Gamma(2)$. Upon relabelling, we can assume

$$\tau = \frac{\int_{\gamma_2} \omega_\lambda}{\int_{\gamma_1} \omega_\lambda} \in \mathbb{H}$$

and its isomorphism class is determined by the value of the modular $\lambda(\tau)$ evaluated at τ, which is invariant under linear transformation by elements in $\Gamma(2)$.

CM cases

Question

How likely is the period ratio $\tau \in \overline{\mathbb{Q}}$?

The answer is not very likely, unless L_λ admits complex multiplication (CM), which is a fundamental result by Schneider.

Theorem (Chowla and Selberg)

If E is an ell. cur. admitting CM by $K = \mathbb{Q}(\sqrt{-d})$ with fund. discriminant $-d$, then all periods of E are algebraic multiples of

$$\omega_{-d} := \Gamma\left(\frac{1}{2}\right) \prod_{0<a<d} \Gamma\left(\frac{a}{d}\right)^{\frac{n\epsilon(a)}{4h_K}}, \tag{14}$$

where n is the number of torsion elements in K, $\epsilon = \left(\frac{-d}{\cdot}\right)$, and h_K is the class number of K

Example

$\omega_{-4} = \Gamma\left(\frac{1}{2}\right) \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$ and $\omega_{-3} = \Gamma\left(\frac{1}{2}\right) \left(\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\right)^{3/2}$

Picard-Fuchs

Question

How is the above picture related to the differential equation $L_{\{1/2,1/2\},\{1,1\}\lambda}F = 0$?

For any $\gamma \in H_1(L_\lambda, \mathbb{Z})$, $p_\gamma(\lambda) := \displaystyle\int_\gamma \omega_\lambda$. If we vary γ analytically, $p_\gamma(\lambda)$ becomes a function of λ. This means by analytic continuation, $\tau(\lambda) = \displaystyle\int_{\gamma_2} \omega_\lambda / \int_{\gamma_1} \omega_\lambda$ is also considered as function of λ.

The differential equation $\mathcal{L}_{\{1/2,1/2\},\{1,1\}\lambda}F = 0$ is called the *Picard-Fuchs equation* of L_λ, by which it means $\mathcal{L}_{\{1/2,1/2\};\{1,1\}}\omega_\lambda$ is an exact form on L_λ.

It follows both $\displaystyle\int_{\gamma_1} \omega_\lambda$ and $\displaystyle\int_{\gamma_2} \omega_\lambda$ are solutions of $\mathcal{L}_{\{1/2,1/2\},\{1,1\};\lambda}F = 0$.

The inverse of the modular lambda function

If we compute $\int_{\gamma_{01}} \omega_\lambda$, up to a scalar, by Euler integral formula, it agrees with $2\int_0^1 \omega_\lambda = 2 \cdot {}_2P_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; \lambda \right] = 2\pi \cdot {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; \lambda \right]$.

The other linearly independent solution is ${}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; 1-\lambda \right]$.

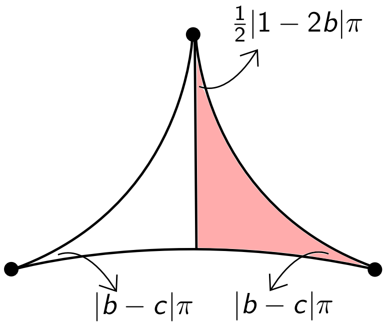
This explains the claim that the Schwarz map $i \dfrac{{}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; 1-\lambda \right]}{{}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; \lambda \right]}$ is the inverse of λ-function.

Hypergeometric formulas: a few techniques:

- From rigidity

$$(1-x)^{-c}\ {}_2F_1\left[\begin{matrix}\frac{1+c}{2}-b & \frac{c}{2} \\ & c-b+1\end{matrix};\ \frac{-4x}{(1-x)^2}\right]$$

$$= {}_2F_1\left[\begin{matrix}b & c \\ & c-b+1\end{matrix};\ x\right].$$



Hypergeometric formulas: a few techniques:

- From the definition/setup

$$
{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; 1\right] = \frac{1}{B(b, c-b)} \int_0^1 (1-t)^{-a} t^{b-1} (1-t)^{c-b-1} dt
$$

$$
= \frac{B(b, c-a-b)}{B(b, c-b)} = \Gamma\left(\frac{c, c-a-b}{c-a, c-b}\right).
$$

- From comparing coefficients

$$
{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; x\right] = (1-x)^{c-a-b}\, {}_2F_1\left[\begin{matrix} c-a & c-b \\ & c \end{matrix}; x\right].
$$

$$
{}_3F_2\left[\begin{matrix} a & b & -n \\ & c & 1+a+b-n-c \end{matrix}; 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}. \tag{15}
$$

- From specializing values in known identities

$$(1-x)^{-c}\,{}_2F_1\left[\begin{matrix}\frac{1+c}{2}-b & \frac{c}{2} \\ & c-b+1\end{matrix};\ \frac{-4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix}b & c \\ & c-b+1\end{matrix};\ x\right].$$

Plotting $x=1$ and use Gauss evaluation leads to

$${}_2F_1\left[\begin{matrix}b & c \\ & c-b+1\end{matrix};\ -1\right] = 2^{-c}\,{}_2F_1\left[\begin{matrix}\frac{1+c}{2}-b & \frac{c}{2} \\ & c-b+1\end{matrix};\ 1\right]$$
$$= 2^{-c}\Gamma\left(\frac{c-b+1, \frac{1}{2}}{\frac{c+1}{2}, 1+\frac{c}{2}-b}\right) = \Gamma\left(\frac{1+c-b, \frac{c}{2}+1}{1+c, \frac{c}{2}-b+1}\right).$$

- From other methods like Bailey transform
  Below is a formula by Whipple.

$$
\begin{aligned}
{}_7F_6&\left[\begin{matrix} a & 1+\frac{a}{2} & c & d & e & f & g \\ & \frac{a}{2} & 1+a-c & 1+a-d & 1+a-e & 1+a-f & 1+a-g \end{matrix}; 1\right] \\
&= \Gamma\left(\frac{1+a-e, 1+a-f, 1+a-g, 1+a-e-f-g}{1+a, 1+a-f-g, 1+a-e-f, 1+a-e-g}\right) \\
&\quad \times \cdot {}_4F_3\left[\begin{matrix} a & e & f & g \\ & e+f+g-a & 1+a-c & 1+a-d \end{matrix}; 1\right], \qquad (16)
\end{aligned}
$$

when both sides terminate.