Hypergeometric Functions, Character Sums and Applications, Part II

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https://alozano.clas.uconn.edu/hypergeometric

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Plan

Day 1. Hypergeometric functions over $\mathbb{C}$

- 1.I Hypergeometric functions
- 1.II Hypergeometric formulas and Legendre curves

Day 2. Over finite fields

- 2.I Hypergeometric functions over finite fields
- 2.II Point counts over finite fields

Day 3. In Galois perspective

- 3.I Hypergeometric Galois representations
- 3.II Modularity results

Day 4. p-adic hypergeometric functions and supercongruences

- 4.I Dwork unit roots and commutative formal group laws
- 4.II Supercongruences

Goals for today

- Introduce hypergeometric functions over finite field in a way parallel to the classical setting
- With which one can obtain many character sum identities (Greene, Evans-Greene, McCarthy, Fuselier-Long-Ramakrishna-Swisher-Tu, $\cdots$)
- Relate hypergeometric character sums to point counts on algebraic varieties

Key properties of the Gamma function

For $\mathrm{Re}(x) > 0$,

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t.$$

Reflection formula

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}, \quad \forall a \in \mathbb{C}$$

Multiplication formula

$$\Gamma(2a)(2\pi)^{1/2} = 2^{2a-\frac{1}{2}}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right), \quad \forall a \in \mathbb{C}. \tag{1}$$

$$(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n. \tag{2}$$

Notation for finite fields

- p: an odd prime
- $\mathbb{F}_q$: a finite field of size q, where $q = p^e$
- $\mathbb{F}_q^\times$: a multiplicative cyclic group of order $q-1$
- $\widehat{\mathbb{F}_q^\times} := \{\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times\} = \langle \omega \rangle$
- $\varepsilon :=$ the trivial character in $\widehat{\mathbb{F}_q^\times}$
- $\phi :=$ the quadratic character in $\widehat{\mathbb{F}_q^\times}$
- $\overline{\chi}$ the inverse of χ
- $\chi(0) = 0$, including $\varepsilon(0) = 0$
- $\Phi(x) := \zeta_p^{\mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(x)}$, ζ_p a fixed pth root of unity

Notation for finite fields

- $$\delta(\chi) := \delta_{\varepsilon}(\chi) := \begin{cases} 1 & \text{if } \chi = \varepsilon, \\ 0 & \text{if } \chi \neq \varepsilon; \end{cases}$$

- $$\delta(x) := \delta_0(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

The orthogonal properties:

- $$\sum_{x \in \mathbb{F}_q} \Phi(x) = 0.$$

- For $\chi, \varphi \in \widehat{\mathbb{F}_q^{\times}}$

  $$\langle \chi, \varphi \rangle = \frac{1}{q-1} \sum_{x \in \mathbb{F}_q^{\times}} \chi(x)\overline{\varphi}(x) = \delta(\chi\overline{\varphi}).$$

Finite Fourier analysis

Any function $f : \mathbb{F}_q \to \mathbb{C}$, it can be expressed as

$$f(x) = f(0)\delta_0(x) + C_\chi \cdot \chi(x),$$

where

$$C_\chi = \langle f, \chi \rangle = \frac{1}{q-1} \sum_{x \in \mathbb{F}_q^\times} f(x)\overline{\chi}(x).$$

Gauss sums

Given $A \in \widehat{\mathbb{F}_q^\times}$, we define its *Gauss sum* by

$$\mathfrak{g}(A) := \sum_{x \in \mathbb{F}_q^\times} A(x)\Phi(x), \quad \Phi(x) := \zeta_p^{\mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(x)}$$

$$\mathfrak{g}(\varepsilon) = \sum_{x \in \mathbb{F}_q^\times} \Phi(x) = -1.$$

Reflection formula

$$\mathfrak{g}(A)\mathfrak{g}(\overline{A}) = qA(-1) - (q-1)\delta(A)$$

$$|\mathfrak{g}(A)| = \sqrt{q}, \text{ if } A \neq \varepsilon$$

Theorem (Hasse-Davenport Relation)

Let $m \in \mathbb{N}$ and $q \equiv 1 \pmod{m}$. For any $\psi \in \widehat{\mathbb{F}_q^\times}$, we have

$$\prod_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi^m = \varepsilon}} \mathfrak{g}(\chi\psi) = -\mathfrak{g}(\psi^m)\psi(m^{-m}) \prod_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi^m = \varepsilon}} \mathfrak{g}(\chi).$$

Theorem (Yamamoto)

Let $M \geq 4$ be an even integer, $p \equiv 1 \pmod{M}$ be a prime, then the reflection formula and the multiplication formulas by divisors of M are the ONLY two types of relations connecting the Gauss sums $\mathfrak{g}(\chi)$ for $\chi \in \widehat{\mathbb{F}_p^\times}$ satisfying $\chi^M = \varepsilon$, when considered as ideals in the ring of algebraic integers.

Jacobi sums

For $A, B \in \widehat{\mathbb{F}_q^\times}$, their *Jacobi sum* is defined as

$$J(A,B) := \sum_{x \in \mathbb{F}_q} A(x)B(1-x).$$

In relation to Gauss sums

$$J(A,B) = \frac{\mathfrak{g}(A)\mathfrak{g}(B)}{\mathfrak{g}(AB)} + (q-1)B(-1)\delta(AB).$$

When $A, B, AB \neq \varepsilon$, $|J(A,B)| = \sqrt{q}$.

Notation:

$$\binom{A}{B} := -B(-1)J(A, \overline{B})$$

Lemma

For any $A \in \widehat{\mathbb{F}_q^\times}$ and $x \in \mathbb{F}_q$, we have

$$\overline{A}(1-x) = \delta(x) + \frac{1}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} J(A\chi, \overline{\chi})\chi(-x)$$

$$= \delta(x) + \frac{-1}{q-1} \sum_{\chi} \binom{A\chi}{\chi} \chi(x).$$

In comparison,

$$(1-x)^{-a} = \sum_{k \geq 0} \frac{(a)_k}{k!} x^k.$$

An analog between the complex and finite field settings

$\frac{1}{N}$	$\to$	an order N character $\eta_N \in \widehat{\mathbb{F}_q^\times}$
$a = \frac{i}{N},\ b = \frac{j}{N}$	$\to$	$A, B \in \widehat{\mathbb{F}_q^\times},\ \ A = \eta_N^i,\ B = \eta_N^j$
x^a	$\to$	$A(x)$
x^{a+b}	$\to$	$A(x)B(x) = AB(x)$
$-a$	$\to$	$\overline{A}$
$\Gamma(a)$	$\to$	$\mathfrak{g}(A)$
$(a)_n = \Gamma(a+n)/\Gamma(a)$	$\to$	$(A)_\chi = \mathfrak{g}(A\chi)/\mathfrak{g}(A)$
$B(a,b)$	$\to$	$J(A,B)$
$\int_0^1 \quad dx$	$\to$	$\sum_{x\in\mathbb{F}}$
$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi},\ \ a \notin \mathbb{Z}$	$\to$	$\mathfrak{g}(A)\mathfrak{g}(\overline{A}) = A(-1)q,\ \ A \neq \varepsilon$
$(ma)_{mn} = m^{mn}\prod_{i=1}^{m}\left(a+\frac{i}{m}\right)_n$	$\to$	$(A^m)_{\psi^m} = \psi(m^m)\prod_{i=1}^{m}(A\eta_m^i)_\psi$

(Vertical) Hasse-Davenport relation

Let $\mathbb{F}_q$ be a finite field,
$\mathbb{F}_{q^r}$ be its degree-r field extension
Let $\mathrm{Tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^r}}$ and $\mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^r}}$ be the trace and norm maps from $\mathbb{F}_{q^r}$ to $\mathbb{F}_q$.
Then for any $\chi \in \widehat{\mathbb{F}_q^\times}$, $\chi_r(x) = \chi(N_{\mathbb{F}_q}^{\mathbb{F}_{q^r}}(x))$ is a multiplicative character for $\mathbb{F}_{q^r}^\times$.

Theorem (Hasse and Davenport)

Notation as above. Let $\mathfrak{g}(\chi)$ be the Gauss sum of χ in $\mathbb{F}_q$, and $\mathfrak{g}(\chi_r)$ be the Gauss sum of χ_r in $\mathbb{F}_{q^r}$. Then,

$$-\mathfrak{g}(\chi_r) = (-\mathfrak{g}(\chi))^r. \tag{3}$$

Hypergeometric functions over finite fields

- Katz, Beukers-Cohen-Mellit, Hoffman-Tu, $\cdots$
- Greene, Evans, Ono, Ahlgren, McCarthy, $\cdots$
- Today's lecture is largely based on a variation of Greene

  Hypergeometric functions over finite fields,
  Jenny G. Fuselier, Ling Long, Ravi Ramakrishna,
  Holly Swisher, and Fang-Ting Tu,
  arxiv:1510.02575 , Memoirs of the AMS, to appear.

Period functions over $\mathbb{C}$

Let

$$ {}_1P_0\left[a_1\ ;\ z\right] := (1-z)^{-a_1} $$

Inductively

$$ {}_{n+1}P_n\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_{n+1} \\ & b_2 & \cdots & b_{n+1} \end{array} ; z\right] := $$

$$ \int_0^1 t^{a_{n+1}-1}(1-t)^{b_{n+1}-a_{n+1}-1}\, {}_nP_{n-1}\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ & b_2 & \cdots & b_n \end{array} ; zt\right] \mathrm{d}t. \tag{4} $$

The order of the a_i's (resp. b_j's) matters.
So

$$ {}_2P_1\left[\begin{array}{cc} a_1 & a_2 \\ & b_2 \end{array} ;\ z\right] = \int_0^1 x^{a_2-1}(1-x)^{b_2-a_2-1}(1-zx)^{-a_1}\,dx. $$

Finite field period functions, defined by induction

Let

$$ {}_1\mathbb{P}_0[A; \lambda; q] := \overline{A}(1-\lambda), $$

for $A \in \widehat{\mathbb{F}_q^\times}$ and $\lambda \in \mathbb{F}_q$. We inductively define

$$ {}_{n+1}\mathbb{P}_n \left[\begin{matrix} A_1 & A_2 & \dots & A_{n+1} \\ & B_2 & \dots & B_{n+1} \end{matrix}; \lambda; q\right] := $$

$$ \sum_{y \in \mathbb{F}_q} A_{n+1}(y) \overline{A}_{n+1} B_{n+1}(1-y) \cdot {}_n\mathbb{P}_{n-1} \left[\begin{matrix} A_1 & A_2 & \dots & A_n \\ & B_2 & \dots & B_n \end{matrix}; \lambda y; q\right], \tag{5} $$

we will drop q if no ambiguity will arise.

When $n=1$ we have,

$$
\begin{aligned}
{}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\ ;\ \lambda\right] &= \sum_{x\in\mathbb{F}_q} B(x)\overline{B}C(1-x)\overline{A}(1-\lambda x). \\
&= \begin{cases} J(B, C\overline{B}), & \text{if } \lambda = 0 \\ \dfrac{B(-1)}{q-1}\displaystyle\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} J(A\chi,\overline{\chi})J(B\chi,\overline{C\chi})\chi(\lambda), & \text{if } \lambda \neq 0 \end{cases} \\
&= \begin{cases} J(B, C\overline{B}), & \text{if } \lambda = 0 \\ \dfrac{BC(-1)}{q-1}\displaystyle\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(\lambda), & \text{if } \lambda \neq 0. \end{cases}
\end{aligned}
$$

Normalization

Define

$$
{}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\ \lambda\right] := \frac{1}{J(B, C\overline{B})}\,{}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\ \lambda\right]. \tag{6}
$$

The ${}_2\mathbb{F}_1$ function satisfies

1) ${}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\ 0\right] = 1;$

2) ${}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\ \lambda\right] = {}_2\mathbb{F}_1\left[\begin{matrix} B & A \\ & C \end{matrix}\,;\ \lambda\right]$, if A, $B \neq \varepsilon$, and A, $B \neq C$.

More generally,

Just like

$$
{}_{n+1}F_n\left[\begin{matrix} a_1 & a_2 & \cdots & a_{n+1} \\ & b_2 & \cdots & b_{n+1} \end{matrix} ;\ z\right]
$$

$$
= \prod_{i=2}^{n+1} B(a_i, b_i - a_i)^{-1} \cdot {}_{n+1}P_n\left[\begin{matrix} a_1 & a_2 & \cdots & a_{n+1} \\ & b_2 & \cdots & b_{n+1} \end{matrix} ; z\right].
$$

We define

$$
{}_{n+1}\mathbb{F}_n\left[\begin{matrix} A_1 & A_2 & \cdots & A_{n+1} \\ & B_2 & \cdots & B_{n+1} \end{matrix} ;\ \lambda\right]
$$

$$
:= \frac{1}{\prod_{i=2}^{n+1} J(A_i, B_i\overline{A_i})} {}_{n+1}\mathbb{P}_n\left[\begin{matrix} A_1 & A_2 & \dots & A_{n+1} \\ & B_2 & \dots & B_{n+1} \end{matrix} ; \lambda\right].
$$

The Legendre curves

For the Legendre curves

$$L_\lambda : \quad y^2 = x(1-x)(1-\lambda x).$$

Let p be a fixed odd prime. If $\lambda \in \mathbb{F}_p$.

$$\begin{aligned}\#(L_\lambda/\mathbb{F}_p) &= 1 + \sum_{x \in \mathbb{F}_p} (1 + \phi(x(1-x)(1-\lambda x))) \\ &= 1 + p + {}_2\mathbb{P}_1 \left[\begin{matrix} \phi & \phi \\ & \varepsilon \end{matrix} ;\ \lambda \right].\end{aligned}$$

The error term

$$a_p(L_\lambda) := (1+p) - \#(L_\lambda/\mathbb{F}_p) = -\,{}_2\mathbb{P}_1 \left[\begin{matrix} \phi & \phi \\ & \varepsilon \end{matrix} ;\ \lambda \right].$$

Hypergeometric algebraic varieties

$$X_\lambda : \quad y^N = x_1^{i_1} \cdots x_n^{i_n} \cdot (1 - x_1)^{j_1} \cdots (1 - x_n)^{j_n} \cdot (1 - \lambda x_1 \cdots x_n)^k.$$

Theorem (Fuselier, Long, Ramakrishna, Swisher, Tu)

Let $q = p^e \equiv 1 \pmod N$ be a prime power, and $\eta_N \in \widehat{\mathbb{F}_q^\times}$ a primitive order N character. Then

$$\#X_\lambda(\mathbb{F}_q) = 1 + q^n + \sum_{m=1}^{N-1} {}_{n+1}\mathbb{P}_n \left[\begin{matrix} \eta_N^{-mk} & \eta_N^{mi_n} & \cdots & \eta_N^{mi_1} \\ & \eta_N^{mi_n+mj_n} & \cdots & \eta_N^{mi_1+mj_1} \end{matrix} ; \lambda; q \right].$$

Hypergeometric formulas over finite fields

We mentioned a few techniques for obtaining hypergeometric formulas such as from definition/setup, from comparing coefficients, from specializing values. Many of them can be applied to the finite field settings to obtain very analogous formulas. *The main technicality lies in analyzing the delta terms.*

Theorem (Gauss summation formula)

- *Over* $\mathbb{C}$.

$$ {}_2P_1\left[\begin{matrix} a & b \\ & c \end{matrix};\ 1\right] = B(b, c-b-a). $$

- *Over* $\mathbb{F}_q$. *For* $A,\ B,\ C \in \widehat{\mathbb{F}_q^{\times}}$,

$$ {}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix};\ 1\right] = J(B, C\overline{AB}). $$

Theorem (Kummer transformation)

- *Over* $\mathbb{C}$.

$$
{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\,;\ x\right] = (1-x)^{c-a-b}\,{}_2F_1\left[\begin{matrix} c-a & c-b \\ & c \end{matrix}\,;\ x\right].
$$

- *Over* $\mathbb{F}_q$.

$$
{}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\ \lambda\right] = C\overline{AB}(1-\lambda)\ {}_2\mathbb{F}_1\left[\begin{matrix} C\overline{A} & C\overline{B} \\ & C \end{matrix}\,;\ \lambda\right] + \delta(1-\lambda)\frac{J(B, C\overline{AB})}{J(C, C\overline{B})}.
$$

Theorem (Pfaff-Saalschütz Evaluation)

- *Over $\mathbb{C}$. For $n \in \mathbb{Z}_{>0}$,*

$$
{}_3F_2\left[\begin{matrix} a & b & -n \\ & d & 1+a+b-d-n \end{matrix} ;\ 1\right] = \frac{(d-a)_n(d-b)_n}{(d)_n(d-a-b)_n}.
$$

- *Over $\mathbb{F}_q$. For any characters A, B, C, $D \in \widehat{\mathbb{F}_q^\times}$, we have*

$$
\begin{aligned}
{}_3\mathbb{P}_2\left[\begin{matrix} A & B & C \\ & D & ABC\overline{D} \end{matrix} ;\ 1\right] &= J(\overline{BC}D, B)J(C, A\overline{D}) - J(D\overline{B}, AB\overline{D}) \\
&= B(-1)J(C, A\overline{D})J(B, C\overline{D}) - BD(-1)J(D\overline{B}, \overline{A}).
\end{aligned} \tag{7}
$$

Theorem (Kummer evaluation formula)

- *Over* $\mathbb{C}$.

$$ {}_2F_1\left[\begin{matrix} b & c \\ & c-b+1 \end{matrix}\,;\,-1\right] = \frac{\Gamma(1+c-b)\Gamma(1+c/2)}{\Gamma(1+c)\Gamma(1-b+c/2)} $$

- *Over* $\mathbb{F}_q$.

$$ {}_2\mathbb{P}_1\left[\begin{matrix} B & C \\ & C\overline{B} \end{matrix}\,;\,-1\right] = \begin{cases} B(-1)J(B\overline{D},\overline{B}) + B(-1)J(\phi B\overline{D},\overline{B}) & \text{if } C = D^2; \\ 0 & \text{otherwise} \end{cases} $$

When $C = D^2$*, further assume "primitive",*

$$ {}_2\mathbb{F}_1\left[\begin{matrix} B & C \\ & C\overline{B} \end{matrix}\,;\,-1\right] = \frac{\mathfrak{g}(C\overline{B})\mathfrak{g}(D)}{\mathfrak{g}(C)\mathfrak{g}(D\overline{B})} + \frac{\mathfrak{g}(C\overline{B})\mathfrak{g}(\phi D)}{\mathfrak{g}(C)g(\phi D\overline{B})}. $$

Do not take the similarities for granted

- Over $\mathbb{C}$.

$$
{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix} ;\ z\right] = \Gamma\left(\frac{c, c-a-b}{c-a, c-b}\right) {}_2F_1\left[\begin{matrix} a & b \\ & a+b+1-c \end{matrix} ;\ 1-z\right]
$$

$$
+\Gamma\left(\frac{c, a+b-c}{a, b}\right)(1-z)^{c-a-b} {}_2F_1\left[\begin{matrix} c-a & c-b \\ & 1+c-a-b \end{matrix} ;\ 1-z\right],
$$

- Over $\mathbb{F}_q$.

$$
{}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix} ;\ \lambda\right] = B(-1)\, {}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & AB\overline{C} \end{matrix} ;\ 1-\lambda\right].
$$

If one looks for the finite field analogue of the mirror map

$$
i\frac{{}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ;\ 1-\lambda\right]}{{}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ;\ \lambda\right]},
$$

the outcome is disappointing.

Another example

- Over $\mathbb{C}$.

$$
{}_2F_1\left[\begin{matrix} a & a-\frac{1}{2} \\ & 2a \end{matrix}\,;\ z\right] = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a}.
$$

- Over $\mathbb{F}_q$. When $z \neq 0$,

$$
\begin{aligned}
&{}_2\mathbb{F}_1\left[\begin{matrix} A & A\phi \\ & A^2 \end{matrix}\,;\ z\right] \\
&= \begin{cases} 0 & \text{if } \phi(1-z) = -1 \\ \left(\overline{A}^2\left(\frac{1+\sqrt{1-z}}{2}\right) + \overline{A}^2\left(\frac{1-\sqrt{1-z}}{2}\right)\right) & \text{if } \phi(1-z) = 1 \end{cases}
\end{aligned}
$$

Remark

While ${}_nF_{n-1}$ is a single function (solution), ${}_n\mathbb{F}_{n-1}$ is by nature an average (trace) function.

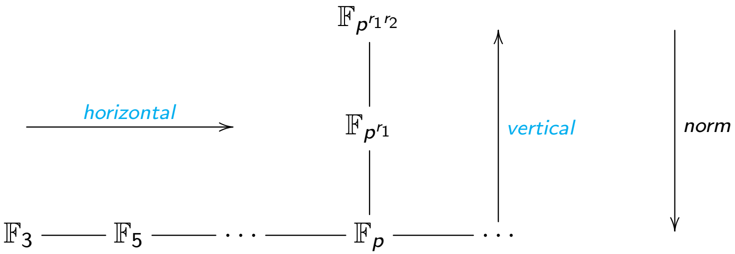
Two views of hypergeometric character sums

For example, if $\alpha = \{\frac{1}{2}, \frac{1}{2}\}, \beta = \{1, 1\}$, use

$$\frac{1}{2} \to \phi \in \widehat{\mathbb{F}_p^{\times}}$$

$${}_2\mathbb{P}_1 \left[\begin{matrix} \phi & \phi \\ & \varepsilon \end{matrix} ;\ \lambda; p \right] = \sum_{x \in \mathbb{F}_p} \phi(x(1-x)(1-\lambda x))$$

in which p can be varied among all odd primes, referred to as a "horizontal" variation.



Vertical view

For a fixed prime p, we can also vary the character sums "vertically" by considering their finite extensions to $\mathbb{F}_{p^r}$ via the norm maps. Vertical variation can be put together as follows.

$$Z(\alpha, \beta; \lambda; p; T) := \exp\left(\sum_{r\geq 1} {}_2\mathbb{P}_1\begin{bmatrix}\phi & \phi \\ & \varepsilon\end{bmatrix}; \ \lambda; p^r\right] \frac{T^r}{r}\right)$$

Recall

$$-\log(1 - aT) = aT + a^2\frac{T^2}{2} + \frac{1}{3}a^3\frac{T^3}{3} + \cdots .$$

Example

When $\lambda = -1$ and $p \equiv 1 \mod 4$, let η_4 be an order-4 character of $\widehat{\mathbb{F}_p^\times}$ and $\eta_{4,r} := \eta_4 \circ N_{\mathbb{F}_p}^{\mathbb{F}_{p^r}}$. By Kummer evaluation formula and the vertical Hasse-Davenport relation,

$$
\begin{aligned}
Z(\alpha, \beta; p; T) &= \exp\left(\phi(-1)\sum_{r\geq 1}(J(\eta_{4,r}, \phi) + J_r(\overline{\eta_{4,r}}, \phi))\frac{T^r}{r}\right) \\
&= \exp\left(\phi(-1)\sum_{r\geq 1}(-1)^{r-1}(J(\eta_4, \phi)^r + J(\overline{\eta_4}, \phi)^r)\frac{T^r}{r}\right) \\
&= (1-\mu_p T)(1-\overline{\mu_p}T) = (1-\mu_p T)(1-p/\mu_p T)
\end{aligned}
$$

where $\mu_p = -\phi(-1)J(\eta_4, \phi) = -J(\eta_4, \phi)$.

Another formulation and comparison

Assume $\lambda \in \mathbb{Q}$. Following McCarthy, Beukers-Cohen-Mellit, when $lcd(\alpha, \beta; \lambda) \mid q-1$, in which case $(q-1)a_j, (q-1)b_j \in \mathbb{Z}$ for all j,

$$H_q(\alpha, \beta; \lambda; \omega)$$
$$:= \frac{1}{1-q} \sum_{k=0}^{q-2} \prod_{j=1}^{n} \frac{\mathfrak{g}(\omega^{k+(q-1)a_j})\mathfrak{g}(\omega^{-k-(q-1)b_j})}{\mathfrak{g}(\omega^{(q-1)a_j})\mathfrak{g}(\omega^{-(q-1)b_j})} \omega^k((-1)^n \lambda).$$

When α, β form a primitive pair,

$$H_q(\alpha, \beta; \lambda; \omega) = {}_n\mathbb{F}_{n-1} \left[\begin{matrix} \omega^{k+(q-1)a_1} & \omega^{k+(q-1)a_2} & \dots & \omega^{k+(q-1)a_n} \\ & \omega^{k+(q-1)b_2} & \dots & \omega^{k+(q-1)b_n} \end{matrix} \; ; \; \lambda \right].$$

If α, β are defined over $\mathbb{Q}$,

Example

For $HD = \{\alpha = \{\frac{1}{3}, \frac{2}{3}\}, \beta = \{1, 1\}; \lambda\}$,

$$(\frac{1}{3})_k(\frac{2}{3})_k(1)_k = 3^{-3k}(1)_{3k}$$

$$\begin{aligned}
{}_2F_1\left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ & 1 \end{matrix} ;\ \lambda\right] &= \sum_{k\geq 0} \frac{(\frac{1}{3})_k(\frac{2}{3})_k}{k!^2}\lambda^k \\
&= \sum_{k\geq 0} \frac{(1)_{3k}}{(1)_k^3}(\lambda/27)^k \\
&= \sum_{k\geq 0} \binom{3k}{k, k, k}(3^{-3}\lambda)^k
\end{aligned}$$

If α, β are defined over $\mathbb{Q}$,

Write

$$\prod_{j=1}^{n} \frac{X - e^{2\pi i a_j}}{X - e^{2\pi i b_j}} = \frac{\prod_{j=1}^{r}(X^{p_j} - 1)}{\prod_{k=1}^{s}(X^{q_k} - 1)}$$

where $p_j, q_k \in \mathbb{Z}_{>0}$ and $p_j \neq q_k$ for all j, k.

Example

For $HD = \{\alpha = \{\frac{1}{3}, \frac{2}{3}\}, \beta = \{1, 1\}; \lambda\}$,

If α, β are defined over $\mathbb{Q}$,

$$
\begin{aligned}
&H_q(\alpha,\beta;\lambda)\\
&\quad := \frac{1}{1-q}\sum_{k=0}^{q-2}\prod_{j=1}^{n}\frac{\mathfrak{g}(\omega^{k+(q-1)a_j})\mathfrak{g}(\omega^{-k-(q-1)b_j})}{\mathfrak{g}(\omega^{(q-1)a_j})\mathfrak{g}(\omega^{-(q-1)b_j})}\,\omega^k((-1)^n\lambda)\\
&:= \frac{(-1)^{r+s}}{1-q}\sum_{m=0}^{q-2} q^{-s(0)+s(m)}\prod_{j=1}^{r}\mathfrak{g}(\omega^{mp_j})\prod_{k=1}^{s}\mathfrak{g}(\omega^{-mq_k})\omega(\epsilon N^{-1}\lambda),
\end{aligned}
$$

where $\epsilon = (-1)^{q_1+\cdots+q_s}$

$$N := N(\alpha,\beta) = \frac{p_1^{p_1}\cdots p_r^{p_r}}{q_1^{q_1}\cdots q_s^{q_s}}$$

$s(m)$ is the multiplicity of $X - e^{2\pi i\frac{m}{q-1}}$ in

$$gcd(\prod_{j=1}^{r}(X^{p_j}-1), \prod_{k=1}^{s}(X^{q_k}-1)).$$

Example

For $HD = \{\alpha = \{\frac{1}{3}, \frac{2}{3}\}, \beta = \{1, 1\}; \lambda\}$,
$r = 1, s = 3, p_1 = 3, q_1 = q_2 = q_3 = 1$, $\epsilon = -1$, $N = 3^3$.

$$H_q(HD_1) = \frac{1}{1-q} \left(1 + \frac{1}{q} \sum_{k=1}^{q-2} \mathfrak{g}(\omega^{3k}) \mathfrak{g}(\omega^{-k})^3 \, \omega^k (-3^{-3}\lambda) \right). \quad (8)$$

In comparison,

$${}_2F_1 \left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ & 1 \end{matrix} ; \ \lambda \right] = \sum_{k \geq 0} \binom{3k}{k, k, k} (3^{-3}\lambda)^k$$

Finite character sums defined over $\mathbb{Q}$

$$H_q(\alpha, \beta; \lambda)$$
$$:= \frac{(-1)^{r+s}}{1-q} \sum_{m=0}^{q-2} q^{-s(0)+s(m)} \prod_{j=1}^{r} \mathfrak{g}(\omega^{mp_j}) \prod_{k=1}^{s} \mathfrak{g}(\omega^{-mq_k}) \omega(\epsilon N^{-1}\lambda),$$

Question

It is now defined for all prime powers coprime to lcd(α, β, λ). But is it a natural or geometric extension?

Let

$$V_{\alpha,\beta}(\lambda): \quad x_1+\cdots+x_r-y_1-\cdots-y_s=0, \quad N\lambda x_1^{p_1}\cdots x_r^{p_r}=y_1^{q_1}\cdots y_s^{q_s},$$

Example

For $\alpha=\{\frac{1}{3},\frac{2}{3}\}, \beta=\{1,1\},\ r=1, s=3, p_1=3, q_1=q_2=q_3=1$. So the corresponding model is

$$X_1-Y_1-Y_2-Y_3=0, \quad 27\lambda X_1^3=Y_1Y_2Y_3.$$

H_q-function and point counts

$$V_{\alpha,\beta}(\lambda): \quad x_1+\cdots+x_r-y_1-\cdots-y_s=0, \quad N\lambda x_1^{p_1}\cdots x_r^{p_r}=y_1^{q_1}\cdots y_s^{q_s},$$

Theorem (Beukers, Cohen, Mellit)

Let the notation p_i, q_j, N as above. Suppose the greatest common denominators of $p_1, \cdots, p_r, q_1, \cdots, q_s$ is one and suppose $N\lambda \neq 1$. Then exists a suitable non-singular completion of $V_{\alpha,\beta}(\lambda)$, denoted by $\overline{V_{\alpha,\beta}(\lambda)}$ such that

$$\#\overline{V_{\alpha,\beta}(\lambda)}/\mathbb{F}_q = P_{rs}(q) + (-1)^{r+s-1} q^{\min(r-1,s-1)} H_q(\alpha,\beta;N\lambda),$$

where

$$p_{rs}(q) = \sum_{m=0}^{\min(r-1,s-1)} \binom{r-1}{m}\binom{s-1}{m}\frac{q^{r+s-m-2}-q^m}{q-1}.$$

Question

Are there other 1-parameter families of algebraic varieties whose Picard-Fuchs equations are hypergeometric?

In the study of Mirror Symmetries originated in String theory, there are families of Calabi-Yau manifolds whose Picard-Fuchs equations are hypergeometric.

Example

$$V_{\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\}}(\psi): \quad X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 - 5\psi\, X_1X_2X_3X_4X_5 = 0$$

$$\mathcal{L}_{\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\},\{1,1,1,1\},\psi^{-5}} F = 0$$

Example

$$V_{\{\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\}}(\psi): \quad \begin{aligned} X_1^3 + X_2^3 + X_3^3 - 3\psi X_4X_5X_6 = 0 \\ X_4^3 + X_5^3 + X_6^3 - 3\psi X_1X_2X_3 = 0 \end{aligned}$$

$$\mathcal{L}_{\{\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\},\{1,1,1,1\},\psi^{-6}} F = 0$$

Dwork family

Degree-N Dwork family of hypersurfacs:

$$\mathcal{D}w_N(\psi): \quad x_0^N + x_1^N + \cdots + x_{N-1}^{N-1} - N\psi x_0 \cdots x_{N-1} = 0.$$

For a fixed ψ, it is a $(N-2)$-dimensional, has a unique up to scalar holomorphic differential $(N-2)$-form. It admits the action of

$$G = \{(a_0, \cdots, a_{N-1}) \in (\mathbb{Z}/\mathbb{NZ})^{N-1}, \sum_{i=0}^{N-1} a_i = 0 \mod N\}$$

via

$$(x_0, \cdots, x_{N-1}) \mapsto (\zeta_N^{a_0} x_0, \cdots, \zeta_n^{a_{N-1}} x_{N-1}).$$

A Picard-Fuchs equation of $Dw_N(\psi)$ is $\mathcal{L}_{\{\frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}\}, \{1, \cdots, 1\}; \psi^{-N}}$.

Question

Will the point counts on $Dw_N(\psi)/\mathbb{F}_q$ be related to the hypergeometric character sums?

Answers from Koblitz, McCarthy.

Hesse pencils

Example

When $N = 3$, it is also known as the Hesse pencil

$$H(\psi): \quad x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2 = 0. \tag{9}$$

When $\lambda = \psi^{-3}$, a morphism from $H(\psi)$ to

$$V_{\{\frac{1}{3},\frac{2}{3}\},\{1,1\}}(\lambda): \quad X_1 - Y_1 - Y_2 - Y_3 = 0, \quad 27\lambda X_1^3 = Y_1 Y_2 Y_3.$$

is given by

$$3\psi x_0 x_1 x_2 \mapsto X_1, \quad x_0^3 \mapsto Y_1, \quad x_1^3 \mapsto Y_2, \quad x_2^3 \mapsto Y_3,$$

which is defined over $\mathbb{Z}$.

Point counts, modulo p

Lemma

Let $p > 3$ be a prime and $\psi \in \mathbb{F}_p$. Let $\mathcal{N}_p(\psi)$ be the number of solutions $H(\psi)$ over $\mathbb{F}_p$. Then

$$\mathcal{N}_p(\psi) \equiv {}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\,;\ \lambda\right]_{p-1} \quad \mathrm{mod}\ p, \quad \textit{where } \lambda = \psi^{-3},$$

the right hand side is a truncated sum.

Proof: Let $f(\underline{x};\psi) = x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2$. Then

$$f(\mathrm{x};\psi)^{p-1} \quad \mathrm{mod}\ p = 1 - \delta_0(f(\mathrm{x})),$$

$$\mathcal{N}_p(\psi) = \sum_{\underline{x}\in\mathbb{F}_p^3} \left(1 - f(\mathrm{x};\psi)^{p-1}\right) = p^3 - \left(\sum_{\underline{x}} f(\mathrm{x};\psi)^{p-1}\right) \quad \mathrm{mod}\ p.$$

Proof continued

Here $f(\underline{x};\psi)^{p-1}$ is a degree $3p-3$ homogeneous polynomial. When summing over all $\mathrm{x} \in \mathbb{F}_p^3$,

$$\sum_{\underline{x}} (x_1x_2x_3)^{p-1} = (p-1)^3 \equiv -1 \mod p,$$

while for all other monomial $\sum_{\underline{x}} x_1^{n_1}x_2^{n_2}x_3^{n_3} = 0 \mod p$. Thus modulo p, $\mathcal{N}(\psi)$ coincides with the coefficient of $(x_1x_2x_3)^{p-1}$, which is

$$\sum_i \binom{p-1}{i,i,i,p-1-3i}(-3\psi)^{p-1-3i} \equiv \sum_{i=0}^{(p-1)/3} \binom{3i}{i,i,i}(3\psi)^{-3i}$$

$$\equiv {}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\,;\ \lambda\right]_{p-1} \mod p.$$

A formal group detour

There is a theory called commutative formal group laws (CFGL). We will illustrate some useful results of Stienstra in the current example. Assume $\psi \in \mathbb{Z}_p \setminus \{0, 1\}$ where $p > 3$.

- Formal. Let b_m be the coefficient of $(xyz)^{m-1}$ of $f(\underline{x}, \psi)^{m-1}$. Putting together

$$\ell(\tau) = \sum_{m \geq 1} b_m \frac{\tau^m}{m}.$$

- Group. Using which we define a group law

$$G(u, v) := \ell^{-1}(\ell(u) + \ell(v)) = u + v + \text{higher degree} \in \mathbb{Z}_p[u, v].$$

  Reproducing the group law of $H(\psi)$ near infinity.

- Law. *Atkin and Swinnerton-Dyer congruences*: for $m, r \geq 1$

$$b_{mp^r} - a_p(H(\psi)) b_{mp^{r-1}} + p b_{mp^{r-2}} \equiv 0 \mod p^r.$$

Back to point counts on $H(\psi)$

The major term is

$$N_q(\psi) := \frac{1}{q(q-1)} \sum_{v \in \mathbb{F}_q, x \in (\mathbb{F}_q^\times)^n} \Phi_q(v(x_1^3 + x_2^3 + x_3^3 - 3\psi x_1 x_2 x_3)).$$

It is a function from $\mathbb{F}_q \mapsto \mathbb{C}$.

Here we use the additive character instead of the delta function so that Gauss sums will appear naturally when we apply finite Fourier analysis.

$$N_q(\psi) = N_q(0)\delta(x) + \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \langle N_q, \chi \rangle \chi(\psi).$$

Computing $\langle N_q, \chi \rangle$ when $\chi \neq \varepsilon$.

$$
\begin{aligned}
(q-1)\langle N_q, \chi \rangle &= \sum_{\psi} N_q(\psi)\bar{\chi}(\psi) \\
&= \frac{1}{q(q-1)} \sum_{\psi} \sum_{v \in \mathbb{F}_q^{\times}, x \in (\mathbb{F}_q^{\times})^n} \Phi_q(vx_1^3)\Phi_q(vx_2^3)\Phi_q(vx_3^3)\Phi_q(-3v\psi x_1x_2x_3) \\
&\qquad \times \chi(\overline{-3v\psi x_1x_2x_3})\chi(-3vx_1x_2x_3) \\
&= \frac{1}{q(q-1)} \mathfrak{g}(\overline{\chi}) \sum_{v \in \mathbb{F}_q^{\times}, x \in (\mathbb{F}_q^{\times})^n} \Phi_q(vx_1^3)\Phi_q(vx_2^3)\Phi_q(vx_3^3)\chi(-3vx_1x_2x_3)
\end{aligned}
\tag{10}
$$

Write

$$
S(\chi) := \sum_{v \in \mathbb{F}_q^{\times}, x \in (\mathbb{F}_q^{\times})^n} \Phi_q(vx_1^3)\Phi_q(vx_2^3)\Phi_q(vx_3^3)\chi(-3vx_1x_2x_3),
$$

which is 0 if χ is not a cubic.

Now we assume $\chi = \eta^3$ is a cube. Then

$$\begin{aligned}\sum_{y \in \mathbb{F}_q^\times} \chi(y)\Phi_q(vy^3) &= \sum_{y \in \mathbb{F}_q^\times} \eta(y^3)\Phi_q(vy^3) \\ &= \sum_{u \in \mathbb{F}_q^\times} \eta(u)\Phi_q(vu)\left(1+\chi_3(u)+\chi_3^2(u)\right) \\ &= \overline{\eta}(v)\mathfrak{g}(\eta) + \overline{\eta\chi}_3(v)\mathfrak{g}(\eta\chi_3) + \overline{\eta}\chi_3(v)\mathfrak{g}(\eta\overline{\chi}_3)\end{aligned}$$

Hence,

$$\begin{aligned}S(\chi) &= \sum_{v \in \mathbb{F}_q^\times} \chi(-3v)\left[\overline{\eta}(v)\mathfrak{g}(\eta) + \overline{\eta\chi}_3(v)\mathfrak{g}(\eta\chi_3) + \overline{\eta}\chi_3(v)\mathfrak{g}(\eta\overline{\chi}_3)\right]^3 \\ &= \sum_{v \in \mathbb{F}_q^\times} \chi(-3)\left[\mathfrak{g}(\eta) + \overline{\chi}_3(v)\mathfrak{g}(\eta\chi_3) + \chi_3(v)\mathfrak{g}(\eta\overline{\chi}_3)\right]^3 \\ &= \chi(-3)\left[\mathfrak{g}(\eta)^3 + \mathfrak{g}(\eta\chi_3)^3 + \mathfrak{g}(\eta\overline{\chi}_3)^3 + 6\mathfrak{g}(\eta)\mathfrak{g}(\eta\chi_3)\mathfrak{g}(\eta\chi_3^2)\right]\sum_{v \in \mathbb{F}_q^\times} 1 \\ &= 6q(q-1)\chi(-1)\mathfrak{g}(\chi) \\ &\quad + \chi(-3)(q-1)\left[\mathfrak{g}(\eta)^3 + \mathfrak{g}(\eta\chi_3)^3 + \mathfrak{g}(\eta\overline{\chi}_3)^3\right]\end{aligned}$$

Putting together,

$$N_q(\psi) = \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \langle N_q, \chi \rangle \chi(\psi) = \frac{1}{3} \sum_{\chi} \langle N_q, \chi^3 \rangle \chi^3(\psi).$$

The major term of the above is

$$\frac{1}{q(q-1)} \sum_{\chi} \chi^3(-3\psi)\mathfrak{g}(\overline{\chi}^3)\mathfrak{g}(\chi)^3.$$

It can be written as $-H_q\left(\left\{\frac{1}{3}, \frac{2}{3}\right\}, \{1,1\}; \frac{1}{\lambda}\right) - \frac{1}{q}$, where $\lambda = \psi^{-3}$.

Arithmetic mirror symmetries

Dwork quintic threefold (when $\psi \neq 0$)

$$V(\psi): \quad X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 - 5\psi\, X_1 X_2 X_3 X_4 X_5 = 0$$

plays an important role in the study of Mirror Symmetries. Its mirror is obtained by the smooth model of $V(\psi)/G$. The Hodge diamonds of $V(\psi)$ and its mirror $\mathcal{V}$ are related by $h^{1,1}(V) = h^{2,1}(\mathcal{V})$, which equals 1 presently.

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Modularity of rigid Calabi-Yau defined over $\mathbb{Q}$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1}=0 & & h^{1,2}=0 & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Theorem (Dieulefait, Gouvêa-Yui)

For any rigid Calabi-Yau X defined over $\mathbb{Q}$ and each prime ℓ, there is a weight 4 modular form f with integer coefficients such that the ℓ-adic Galois representation arising from the third étale cohomology group of X is isomorphic to the ℓ-adic Deligne representation associated to f.

Ordinary Calabi-Yau differential equations

A Picard-Fuchs differential operator of $V(\psi)$ is given using variable $\lambda = \psi^{-5}$

$$\theta^4 - 5^{-4}\lambda(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4), \quad \text{where } \theta := \lambda\frac{d}{d\lambda},$$

whose unique (up to scalar) holomorphic solution near zero is

$$\sum_{k=0}^{\infty} \frac{(5k)!}{k!^5}\,(5^{-5}\lambda)^k = {}_4F_3\left[\begin{matrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \\ & 1 & 1 & 1 \end{matrix}\,;\ \lambda\right].$$

There are 14 such Calabi-Yau differential equations (by Almkvist, van Enckevort, van Straten, Zudilin). Their parameter sets are of the form $\alpha = \{r_1, 1-r_1, r_2, 1-r_2\}, \beta = \{1,1,1,1\}$, where

$$r_1, r_2 \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$$

or

$$(r_1, r_2) \in \{(\frac{1}{5}, \frac{2}{5}), (\frac{1}{8}, \frac{3}{8}), (\frac{1}{10}, \frac{3}{10}), (\frac{1}{12}, \frac{5}{12})\}.$$

The defining equations of the corresponding Calabi-Yau three-folds can be given explicitly.

To compute their Picard-Fuchs equation, there is a general method called the Gel'fand, Zelevinskiĭ, and Kapranov (GKZ) method.

When $\psi = \lambda = 1$, the corresponding Calabi-Yau manifolds are rigid. By modularity theorem of Dieulefait and Gouvêa-Yui, they are all modular.

We will explain to identify the corresponding weight-4 modular forms in the next lecture.

Take-away from this lecture

- Hypergeometric functions over finite fields can be studied in a way parallel to the classical setting
- There are horizontal (more to say in Lecture III) and vertical perspectives
- They are useful for point counts, especially for 1-parameter families of varieties whose Picard-Fuchs equations are hypergeometric
- Some of these varieties are important for the development of mirror symmetries in string theory. In return, the notation of "mirror" gives us a connection between two types of hypergeometric varieties.