# HYPERGEOMETRIC FUNCTIONS, CHARACTER SUMS AND APPLICATIONS

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Abstract. We summarize several aspects of hypergeometric functions based on our recent work [7, 6, 11, 16, 17, 18, 19, 20] and our understanding of the subjects.

## 3. Hypergeometric Galois representations

The goal here is to construct and compute explicit Galois representations from hypergeometric data. With the inputs from classical formulas such as Clausen's formula and Whipple's formulas, we can further obtain explicit information on these hypergeometric Galois representations.

For any finite field $\mathbb{F}_q$ of characteristic $p > 2$, let $\omega$ be a generator of the group $\widehat{\mathbb{F}_q^\times} := \mathrm{Hom}(\mathbb{F}_q^\times, F^\times)$ of multiplicative characters of $\mathbb{F}_q$, where $F = \mathbb{C}$ or $\mathbb{C}_p$. Let $\alpha = \{a_1, \cdots, a_n\}, \beta = \{1, b_2, \cdots, b_n\}$ be two multi-sets with entries in $\mathbb{Q}$. Denote by $M := lcd(\alpha, \beta)$ the least positive common denominator of all $a_i, b_j$. For a finite field $\mathbb{F}_q$ containing a primitive $M$th root of 1 and any $\lambda \in \mathbb{F}_q$, recall that we write

$$\mathbb{P}(\alpha, \beta; \lambda; \mathbb{F}_q; \omega) := {}_n\mathbb{P}_{n-1}\left[\begin{matrix}\omega^{(q-1)a_1} & \omega^{(q-1)a_2} & \dots & \omega^{(q-1)a_n} \\ & \omega^{(q-1)b_2} & \dots & \omega^{(q-1)b_n}\end{matrix}\; ; \; \lambda; q\right].$$

Its value depends on $a_i$ and $b_j$ modulo $\mathbb{Z}$, as well as their orders and the choice of $\omega$. Similarly

$$\mathbb{F}(\alpha, \beta; \lambda; \mathbb{F}_q; \omega) := {}_n\mathbb{F}_{n-1}\left[\begin{matrix}\omega^{(q-1)a_1} & \omega^{(q-1)a_2} & \dots & \omega^{(q-1)a_n} \\ & \omega^{(q-1)b_2} & \dots & \omega^{(q-1)b_n}\end{matrix}\; ; \; \lambda; q\right].$$

### 3.1. **A motivating example.**

We return to the Legendre curves to illustrate the main theorems of this section.

Let $\lambda \in \mathbb{Q} \setminus \{0, 1\}$ be fixed and $\ell$ be a fixed prime number.

$$L_\lambda : \quad y^2 = x(1-x)(1-\lambda x).$$

It is an elliptic curve defined over $\mathbb{Q}$. The torsion points $\varprojlim_n L_\lambda[\ell^n]$ give rise to a continuous representation

$$\rho_{\lambda,\ell} : G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_\ell).$$

For details, see standard textbooks such as [24] by Silverman. For $p \nmid Cond(L_\lambda)$, the conductor of $L_\lambda$.

$$\mathrm{Tr}\rho_{\lambda,\ell}(\mathrm{Frob}_p) = p - \#(L_\lambda/\mathbb{F}_p),$$

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where $\mathrm{Frob}_p$ stands for the Frobenius conjugacy class at $p$. Recall that we can count points on $L_\lambda$ using the period function

$$\#(L_\lambda/\mathbb{F}_p) = \sum_{x\in\mathbb{F}_p}(1+\phi(x(1-x)(1-\lambda x)) = p + \mathbb{P}(\{\frac{1}{2},\frac{1}{2}\},\{1,1\};\lambda;\mathbb{F}_p;\omega).$$

It is independent of the choice of $\omega$ as the quadratic character $\phi$ in $\widehat{\mathbb{F}_p^\times}$ is unique.

Consequently

$$\mathrm{Tr}\rho_{\lambda,\ell}(\mathrm{Frob}_p) = -\mathbb{P}(\{\frac{1}{2},\frac{1}{2}\},\{1,1\};\lambda;\mathbb{F}_p) = \phi(-1)\mathbb{F}(\{\frac{1}{2},\frac{1}{2}\},\{1,1\};\lambda;\mathbb{F}_p). \tag{1}$$

As $L_\lambda$ is an elliptic curve defined over $\mathbb{Q}$, each $\rho_{\lambda,\ell}$ is isomorphic to an $\ell$-adic Galois representation arising from a weight-2 cuspidal Hecke eigenform $f_\lambda$. Thus for each $p \nmid Cond(L_\lambda)$

$$\mathrm{Tr}\rho_{\lambda,\ell}(\mathrm{Frob}_p) = a_p(f_\lambda),$$

the $p$th coefficient of $f_\lambda$. Putting together,

$$\phi(-1)\mathbb{F}(\{\frac{1}{2},\frac{1}{2}\},\{1,1\};\lambda;\mathbb{F}_p) = a_p(f_\lambda). \tag{2}$$

The level of $f_\lambda$ equals the conductor of $L_\lambda$. It only consists of primes dividing the discriminant of $L_\lambda$ in its minimal model defined over $\mathbb{Z}$. More generally,

**Theorem 3.1.** *Given a prime $\ell$ and a 2-dimensional absolutely irreducible representation $\rho$ of $G_\mathbb{Q}$ over $\overline{\mathbb{Q}}_\ell$ that is odd, unramified at almost all primes, and its restriction to a decomposition subgroup $D_\ell$ at $\ell$ is crystalline with Hodge-Tate weight $\{0,r\}$ where $1\le r\le \ell-2$ and $\ell+1\nmid 2r$, then $\rho$ is modular and corresponds to a weight $r+1$ holomorphic Hecke eigenform.*

The actual identification of the target modular form can be also carried out by computing the trace of the representation at various Frobenius conjugacy classes $\mathrm{Frob}_p$. The following theorem of Serre (cf. [23] by Serre or [8, Theorem 2.2] by Dieulefait) is helpful to narrow the search for the corresponding modular forms.

**Theorem 3.2** (Serre)**.** *Let $f$ be an integral weight holomorphic Hecke eigenform with coefficients in $\mathbb{Z}$. Then the $p$-exponents of the level of $f$ are bounded by 8 for $p=2$, by 5 for $p=3$, and by 2 for all other bad primes.*

3.2. **A systematic way to get compatible hypergeometric functions over various finite fields.** For a general hypergeometric parameter set $\alpha,\beta$, the integer $M = lcd(\alpha,\beta)$ may likely be larger than 2. To look for a result generalizing Equation (1) above, we first consider how to vary the finite fields and then the choices of the characters.

We will use the following notation.

- Degree $M := lcd(\alpha,\beta)$
- Field $K := \mathbb{Q}(\zeta_M)$
- Ring $\mathcal{O}_K = \mathbb{Z}[\zeta_M]$
- Group $G(M) := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, an index-$\varphi(N)$ subgroup of $G_\mathbb{Q} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- $\wp$ any nonzero prime ideal of $\mathbb{Z}[\zeta_M, 1/M]$
- $k_\wp := \mathcal{O}_K/\wp$, its size $q(\wp) := |k_\wp| \equiv 1 \mod M$
- $\mathrm{Frob}_\wp$ the Frobenius conjugacy class of $G(M)$ at $\wp$.

Thus we go from nonzero prime ideals in $\mathbb{Z}[\zeta_M, 1/M]$ to finite fields by

$$\wp \mapsto \kappa_\wp = \mathbb{Z}[\zeta_M]/\wp.$$

According to the analogy between the classical and the finite field settings

$$\frac{1}{M} \mapsto \eta_M \in \widehat{\kappa_\wp^\times}, \text{ an primitive order } M \text{ character.}$$

Once $\eta_M$ is fixed, by the inductive formula, $\mathbb{P}(\alpha,\beta;\lambda;\kappa_\wp)$ can be defined accordingly. We now explain to choose $\eta_M$ as $\wp$ varies. This can be done using the $M$th residue symbol, see [11, Definition 5.8]. Namely for a given $\wp$

$$\eta_M(x) := \iota_\wp(\frac{1}{M})(x) \equiv x^{\frac{q(\wp)-1}{M}} \mod \wp, \quad \forall x \in \mathbb{Z}[\zeta_M].$$

Here in a compatible way means the following. We first fix a primitive order $M$ character $\chi_M$ of the multiplicative group consisting of fractional ideals $\mathcal{I}_M$ of $\mathbb{Z}[\zeta_M]$ coprime to $M$, i.e. $\chi_M : \mathcal{I}_M \to \mathbb{C}^\times$. Then for each nonzero prime ideal $\wp$ of $\mathbb{Z}[\zeta_M]$, if $\mathfrak{a},\mathfrak{b} \in \mathcal{I}_M$ and $\mathfrak{a} \equiv \mathfrak{b} \mod \wp$, then $\chi_M(\mathfrak{a}) = \chi_M(\mathfrak{b})$ as $|\kappa_\wp| = 1 \mod M$. Hence we can define a character $\iota_\wp(\frac{1}{M})$ on $\kappa_\wp$ via

$$\iota_\wp(\frac{1}{M})(x\wp) := \chi_M(x), \quad \forall x \in \mathbb{Z}[\zeta_N, 1/M].$$

Another way to proceed is as follows. For a prime ideal $\wp$ of $\mathbb{Z}[\zeta_M, 1/M]$, $\zeta_M \mod \wp$ in the residue field $\kappa_\wp$ of $\wp$ has order $M$, and it generates the cyclic group $(\kappa_\wp^\times)^{(N(\wp)-1)/M}$. Put

$$\mathbb{P}(\alpha,\beta;\lambda;\kappa_\wp) = \mathbb{P}(\alpha,\beta;\lambda;\kappa_\wp;\omega_\wp)$$

where $\omega_\wp$ is a generator of $\widehat{\kappa_\wp^\times}$ so that

$$\omega_\wp(\zeta_M \mod \wp) = \zeta_M^i, \quad i \in (\mathbb{Z}/M\mathbb{Z})^\times. \tag{3}$$

We choose $i=-1$ by default. Note that $\mathbb{P}(\alpha,\beta;\lambda;\kappa_\wp)$ is independent of the choice of $\omega_\wp$, but depends on the choice of $i$ on $\zeta_M^i$.

### 3.3. The $|\alpha| = |\beta| = 2$ case.

$${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix};\, z\right] = \frac{1}{B(b,c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx,$$

(here $\int_0^1$ could be replaced by $\int_{\gamma_{01}}$ with a normalizing factor where $\gamma_{01}$ is the Pochhammer integral as in Section 1) can be considered as a normalized period on

$$C_\lambda^{[N;i,j,k]}: \quad y^N = x^i(1-x)^j(1-\lambda x)^k, \quad \text{where}$$

$$N = \mathrm{lcd}(a,b,c), \quad i = N\cdot(1-b), \quad j = N\cdot(1+b-c), \quad k = N\cdot a.$$

For this subsection we follow the convention of [6] to use $N$ instead of $M$ for $lcd(\alpha,\beta)$. It is a singular curve defined over $\mathbb{Q}(\lambda)$, we use $X_\lambda^{[N;i,j,k]}$ (or simply $X(\lambda)$ when $N,i,j,k$ are fixed) for its smooth model. If $\alpha,\beta$ are primitive, then $N \nmid i,j,k,i+j+k$. We can also assume $0 < i,j,k < N$ up to change variables $x,y$. Note that $C_\lambda^{[N;i,j,k]}$ admits an automorphism

$$\zeta : (x,y) \mapsto (x, \zeta_N^{-1}y). \tag{4}$$

Use $J_\lambda^{[N;i,j,k]}$ to denote the Jacobian of $X_\lambda^{[N;i,j,k]}$. For each proper divisor $d$ of $N$, $J_\lambda^{[N;i,j,k]}$ contains a factor which is isogenous to $J_\lambda^{[d;i,j,k]}$ over $\mathbb{Q}(\lambda,\zeta_N)$. Use $J_\lambda^{\mathrm{prim}}$ to denote the primitive part of $J_\lambda^{[N;i,j,k]}$, which is of dimensional $\varphi(N)$, by Archinard [2].

**Proposition 3.3** (Archinard)**.** *A basis of $H^0(X(\lambda),\Omega^1)$ can be chosen by the regular pull-backs of differentials on $C_\lambda^{[N;i,j,k]}$ of the form*

$$\omega(\lambda) = \frac{x^{c_0}(1-x)^{c_1}(1-\lambda x)^{c_2}dx}{y^n}, \quad \textit{where } 0 \le n \le N-1,\ c_i \in \mathbb{Z}, \tag{5}$$

*satisfying the following conditions equivalent to the pullback of $\omega$ being regular at $0, 1, \frac{1}{\lambda}, \infty$ respectively,*

$$c_0 \geq \frac{ni + \gcd(N,i)}{N} - 1, \quad c_1 \geq \frac{nj + \gcd(N,j)}{N} - 1, \quad c_2 \geq \frac{nk + \gcd(N,k)}{N} - 1,$$

$$c_0 + c_1 + c_2 \leq \frac{n(i+j+k) - \gcd(N, i+j+k)}{N} - 1.$$

Using the induced action of $\zeta$, $H^0\left(X(\lambda), \Omega^1\right)$ decomposes into eigenspaces $V_i$ such that for each $v \in V_i, \zeta^* v = \zeta_N^i v$.

If $\gcd(n,N) = 1$, the dimension of $V_n$ is given by

$$\dim V_n = \left\{\frac{ni}{N}\right\} + \left\{\frac{nj}{N}\right\} + \left\{\frac{nk}{N}\right\} - \left\{\frac{n(i+j+k)}{N}\right\},$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of $x$, see [1] by Archinard. Furthermore,

$$\dim V_n + \dim V_{N-n} = 2.$$

The elements of $V_n$ with $\gcd(n,N) = 1$ are said to be *new* or *primitive*. The subspace

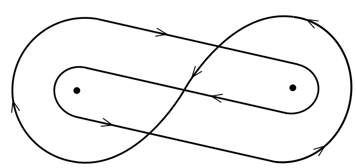
$$H^0(X(\lambda), \Omega^1)^{\mathrm{prim}} = \bigoplus_{\gcd(n,N)=1} V_n$$

is of dimension $\varphi(N)$, Euler's totient function of $N$, see [2].

Under the assumptions that $i,j,k > 0$, $N \nmid i, j, k,\ i+j+k$, by work in [2], $J_\lambda^{\mathrm{prim}}$ is of dimension $\varphi(N)$, and is defined over $\mathbb{Q}(\lambda)$. Moreover the Jacobian variety $J_\lambda^{\mathrm{prim}}$ is isogenous to the complex torus $\mathbb{C}^{\varphi(N)}/\Lambda(\lambda)$, see [28] by Wolfart, where the lattice is

$$\Lambda(\lambda) = \left\{ \left( \sigma_n(u) \int_{\gamma_{01}} \omega_n(\lambda) + \sigma_n(v) \int_{\gamma_{\frac{1}{\lambda}\infty}} \omega_n(\lambda) \right)_{n \in (\mathbb{Z}/N\mathbb{Z})^\times} : u, v \in \mathbb{Z}[\zeta_N] \right\}, \tag{6}$$

when $\int_{\gamma_{01}} \omega_n(\lambda)$ and $\int_{\gamma_{\frac{1}{\lambda}\infty}} \omega_n(\lambda)$ are linearly independent functions of $\lambda$, where $\gamma_{01}$ and $\gamma_{\frac{1}{\lambda}\infty}$ are two Pochhammer contour paths, and $\sigma_n \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ defined by $\zeta_N \mapsto \zeta_N^n$.



*Example* 3.1. For the family $C_\lambda^{[5;3,4,4]}$, according to the previous Proposition, a basis of $H^0\left(X(\lambda), \Omega^1\right)$ is

$$\frac{dx}{y}, \ \frac{xdx}{y}, \ \frac{x(1-x)(1-\lambda x)dx}{y^2}, \ \frac{x(1-x)^2(1-\lambda x)^2 dx}{y^3}.$$

From which we know $\dim V_1 = 2$, $\dim V_2 = 1$, $\dim V_3 = 1$ and $\dim V_4 = 0$.

*Example* 3.2. For the family $C_\lambda^{[6;4,3,1]}$, a basis of $H^0\left(X(\lambda),\Omega^1\right)$ can be given as

$$\omega_1=\frac{dx}{y}=x^{-2/3}(1-x)^{-1/2}(1-\lambda x)^{-1/6}dx$$
$$\omega_4=\frac{x^2(1-x)^2dx}{y^4}=x^{-2/3}(1-\lambda x)^{-2/3}dx$$
$$\omega_5=\frac{x^3(1-x)^2}{y^5}dx=x^{-1/3}(1-x)^{-1/2}(1-\lambda x)^{-5/6}dx.$$

Among them, $\omega_4$ can be considered as a differential 1-form on $C_\lambda^{[3;4,3,1]}$.

We now consider the expressions for $\int_{\gamma_{01}}\omega_n(\lambda)$ and $\int_{\gamma_{\frac{1}{\lambda}\infty}}\omega_n(\lambda)$. Recall that a basis can be chosen such that each element is of the form given in (5). If $\omega(z)=x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx$ then

$$(7)\quad \begin{aligned}\int_{\gamma_{01}}\omega(z)=&(1-e^{2\pi ib})(1-e^{2\pi i(c-b)})\int_0^1\omega(z)\\ =&(1-e^{2\pi ib})(1-e^{2\pi i(c-b)})\,{}_2P_1\left[\begin{matrix}a & b\\ & c\end{matrix};\,z\right]\\ \int_{\gamma_{\frac{1}{z}\infty}}\omega(z)=&(1-e^{-2\pi ia})(1-e^{2\pi i(c-a)})\int_{\frac{1}{z}}^{\infty}\omega(z)\\ =&(1-e^{-2\pi ia})(1-e^{2\pi i(c-a)})(-1)^{c-a-b-1}z^{1-c}(1-z)^{c-a-b}\,{}_2P_1\left[\begin{matrix}1-a & 1-b\\ & 2-c\end{matrix};\,z\right].\end{aligned}$$

**Remark 1.** *If $c=1$, like the Legendre curves case considered in Part I, a different path like $\gamma_{0\infty}$ can be used instead.*

If $N<i+j+k<2N$, then

$$\dim V_n=\dim V_{N-n}=1,\quad \gcd(N,n)=1,$$

and

$$V_n=\langle\omega_n\rangle,\quad \text{where } \omega_n=x^{-\{ni/N\}}(1-x)^{-\{nj/N\}}(1-\lambda x)^{-\{nk/N\}}dx.$$

Thus, when $1\le i,j,k<N$, $\gcd(N,i,j,k)=1$, $N\nmid i+j$ nor $i+j+k$, and $\lambda\neq 0,\,1$, the period lattice $\Lambda(\lambda)$ can be expressed in terms of

$${}_2P_1\left[\begin{matrix}\{\frac{nk}{N}\} & 1-\{\frac{ni}{N}\}\\ & 2-\{\frac{ni}{N}\}-\{\frac{nj}{N}\}\end{matrix};\lambda\right].$$

When $0<i+j+k<N$ or $2N<i+j+k<3N$, i.e. the corresponding triangle group is spherical, we do not have such a general form for the vector space $V_n$.

If $\lambda\in\mathbb{Q}\setminus\{0,1\}$, there is a $2\varphi(N)$-dimensional $\ell$-adic Galois representation $\rho_{\lambda,\ell}$ of $G_{\mathbb{Q}}$ arising from $J_\lambda^{\mathrm{prim}}$. Using the action induced from (4), $\rho_{\lambda,\ell}|_{G(M)}$ decomposes as a direct sum of $\varphi(N)$ copies of 2-dimensional Galois representations.

**Theorem 3.4** (Fuselier, Long, Ramakrishna, Swisher, Tu, [11])**.** *Let $a,b,c\in\mathbb{Q}$ with least common denominator $N$ such that $a$, $b$, $a-c$, $b-c\notin\mathbb{Z}$ and $\lambda\in\mathbb{Q}\setminus\{0,1\}$. Set $K=\mathbb{Q}(\zeta_N)$ and denote its ring of integers $\mathcal{O}_K$. Let $\ell$ be any prime. Then there exists a representation*

$$\sigma_{\lambda,\ell}:G_K:=Gal(\overline{K}/K)\to GL_2(\overline{\mathbb{Q}}_\ell),$$

*depending on a, b and c, that is unramified at all nonzero prime ideals $\wp$ of $\mathbb{Z}[\zeta_N, 1/N\ell]$ and satisfy $ord_\wp(\lambda) = 0 = ord_\wp(1-\lambda)$. Furthermore, the trace of Frobenius at $\wp$ in the image of $\sigma_{\lambda,\ell}$ is the well-defined algebraic integer*

$$-\mathbb{P}(\{a,b\},\{1,c\};\lambda;\kappa_\wp).$$

**Remark 2.** *We use $\overline{\sigma}_{\lambda,\ell}$ to denote its complex conjugate, namely the requirement for the generator as in (3) is changed from $i=-1$ by default to*

$$\omega_\wp(\zeta_M \mod \wp) = \zeta_M.$$

**Question 1.** *Assume that $\varphi(N)=2$. When does $J_\lambda^{prim}$ admit quaternionic multiplication (QM)? By Definition 3.1.1 of [3], $J_\lambda^{prim}$ admits QM means it affords the actions of two linear maps $J_s, J_t$ such that $J_s^2 = J_t^2 = -id$ and $J_sJ_t = -J_sJ_t$. Using Clifford theory [5], if $\sigma_{\lambda,\ell}$ is absolute irreducible, then $\sigma_{\lambda,\ell}$ and $\overline{\sigma}_{\lambda,\ell}$ only differ by a finite order character of $G(M)$, see [3, 15].*

In addition to the motivation we mentioned in Section 1 about the hypergeometric monodromy groups being "triangle groups", and arithmetic Shimura curves parameterize 2-dimensional abelian varieties admitting quaternionic multiplications, we are also motivated by a modularity theorem on 4-dimensional Galois representations of $G_\mathbb{Q}$ admitting QM by Atkin, Li, Liu and Long, see [3, Theorem 3.1.2].

**Theorem 3.5** (Deines, Fuselier, Long, Swisher, Tu)**.** *Let $N = 3,4,6$ and other notations and assumptions as above, in particular, $N \nmid i+j+k, i,j,k$. Then for each $\lambda \in \overline{\mathbb{Q}}$, $End_0(J_\lambda^{prim})$ contains a quaternion algebra over $\mathbb{Q}$ if and only if*

$$B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) \Big/ B\left(\frac{k}{N}, \frac{2N-i-j-k}{N}\right) \in \overline{\mathbb{Q}}.$$

*Idea of the proof.* The traces of $\sigma_{\lambda,\ell}$ are $-{}_2\mathbb{P}_1 = -J\cdot {}_2\mathbb{F}_1$ functions. Due to the Euler transformation (see Equation (30) of Section 1), ${}_2\mathbb{F}_1$ and ${}_2\overline{\mathbb{F}}_1$ in the current context only differ by a finite character. So we want to know when $J/\overline{J}$ is a finite order character. Yamaoto's result (see Theorem 1.2 of Section 2) says if it is the case then the above ratio is algebraic. Conversely, we use a result of Wüstholz. □

**Theorem 3.6** (Wüstholz)**.** *Let $A$ be an abelian variety isogenous over $\overline{\mathbb{Q}}$ to the direct product $A_1^{n_1}\times\cdots\times A_k^{n_k}$ of simple, pairwise non-isogenous abelian varieties $A_\mu$ defined over $\overline{\mathbb{Q}}$, $\mu = 1,\dots,k$. Let $\Lambda_{\overline{\mathbb{Q}}}(A)$ denote the space of all periods of differentials, defined over $\overline{\mathbb{Q}}$, of the first kind and the second on $A$. Then the vector space $\widehat{V}_A$ over $\overline{\mathbb{Q}}$ generated by 1, $2\pi i$, and $\Lambda_{\overline{\mathbb{Q}}}(A)$, has dimension*

$$\dim_{\overline{\mathbb{Q}}} \widehat{V}_A = 2 + 4\sum_{\nu=1}^{k} \frac{\dim A_\nu^2}{\dim_\mathbb{Q}(End_0 A_\nu)},$$

*where $End_0(A_\nu) = End(A_\nu)\otimes_\mathbb{Z}\mathbb{Q}$.*

*Example* 3.3. For the smooth model $X_\lambda := X_\lambda^{[6;4,3,1]}$, the new space $H^0(X(\lambda),\Omega^1)^{\text{prim}}$ is spanned by $\omega_1 = \frac{dx}{y} = x^{-2/3}(1-x)^{-1/2}(1-\lambda x)^{-1/6}dx$, $\omega_5 = \frac{x^3(1-x)^2}{y^5}dx = x^{-1/3}(1-x)^{-1/2}(1-\lambda x)^{-5/6}dx$ as in Example 3.2. The corresponding periods as in Equation (7) are

$$\tau_1 = \int_{\gamma_{01}} \omega_1 \sim B\left(\frac{1}{3},\frac{1}{2}\right) {}_2F_1\left[\begin{matrix}\frac16 & \frac13\\ & \frac56\end{matrix};\ \lambda\right],$$
$$\tau_1' = \int_{\gamma_{1/\lambda\infty}} \omega_1 \sim B\left(\frac{1}{3},\frac{5}{6}\right) {}_2F_1\left[\begin{matrix}\frac12 & \frac13\\ & \frac76\end{matrix};\lambda\right],$$
$$\tau_5 = \int_{\gamma_{01}} \omega_5 \sim B\left(\frac{2}{3},\frac{1}{2}\right) {}_2F_1\left[\begin{matrix}\frac56 & \frac23\\ & \frac76\end{matrix};\lambda\right],$$
$$\tau_5' = \int_{\gamma_{1/\lambda\infty}} \omega_5 \sim \lambda^{-\frac16} B\left(\frac{2}{3},\frac{1}{6}\right) {}_2F_1\left[\begin{matrix}\frac12 & \frac23\\ & \frac56\end{matrix};\lambda\right],$$

where $x \sim y$ means $x = cy$ for some $c \in \overline{\mathbb{Q}}$.

*Exercise* 3.1. 1). Use Euler's transformation formula to verify that

$$\tau_5'/\tau_1 \sim \alpha(\lambda)\frac{\Gamma\left(\frac23\right)\Gamma\left(\frac16\right)}{\Gamma\left(\frac13\right)\Gamma\left(\frac12\right)}, \qquad \tau_1'/\tau_5 \sim \frac{\Gamma\left(\frac13\right)\Gamma\left(\frac56\right)}{\alpha(\lambda)\Gamma\left(\frac23\right)\Gamma\left(\frac12\right)},$$

where $\alpha(\lambda) = (-1)^{\frac23}\lambda^{-\frac16}(1-\lambda)^{-\frac13}$.

2). Use properties of Gamma functions to verify both $\tau_5'/\tau_1, \tau_1'/\tau_5 \in \overline{\mathbb{Q}}$.

The corresponding period matrix of $J_\lambda^{[6;4,3,1],prim}$ is given by

$$\Lambda_\lambda = \left(\begin{array}{cc|cc} \tau_1 & \zeta\tau_1 & \beta_1\tau_5 & \zeta\beta_1\tau_5\\ \tau_5 & \zeta^{-1}\tau_5 & \beta_2\tau_1 & \zeta^{-1}\beta_2\tau_1\end{array}\right)$$

where $\zeta = \zeta_6$ and

$$\tau_1 = {}_2P_1\left[\begin{matrix}\frac16 & \frac13\\ & \frac56\end{matrix};\ \lambda\right], \quad \tau_5 = {}_2P_1\left[\begin{matrix}\frac56 & \frac23\\ & \frac76\end{matrix};\ \lambda\right],$$
$$\beta_1 = (-1)^{-2/3}\left(\lambda^{1/6}(1-\lambda)^{1/3}\sqrt[3]{2}\right), \quad \beta_2 = (-1)^{2/3}\left(\lambda^{-1/6}(1-\lambda)^{-1/3}\sqrt[3]{4}\right),$$

satisfying $\beta_1\beta_2 = 2$. From this, we can see that the endomorphisms

$$\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}, \quad \begin{pmatrix}0 & \beta_1\\ \beta_2 & 0\end{pmatrix}$$

are contained in the endomorphis ring of the lattice $\Lambda_\lambda$, $End_0(\Lambda_\lambda)$. For a generic choice of $\lambda \in \mathbb{Q}$ (non-CM case), these two generate $End_0(J_\lambda^{new})$ which is isomorphic to $\left(\frac{-3,2}{\mathbb{Q}}\right)$.

The traces of the Galois representations $\sigma_{\lambda,\ell}$ and $\overline{\sigma}_{\lambda,\ell}$ in Theorem 3.4 correspond to

$$-\mathbb{P}\left(\left\{\frac16,\frac13\right\},\left\{1,\frac56\right\};\lambda\right), \quad -\mathbb{P}\left(\left\{\frac56,\frac23\right\},\left\{1,\frac76\right\};\lambda\right),$$

respectively. Furthermore, Proposition 2.8 of Part II implies that

$$\frac{\mathbb{P}\left(\left\{\frac56,\frac23\right\},\left\{1,\frac76\right\};\lambda\right)}{\mathbb{P}\left(\left\{\frac16,\frac13\right\},\left\{1,\frac56\right\};\lambda\right)} = \omega_\wp^{-\frac{N(\wp)-1}{6}}(\lambda)\omega_\wp^{-\frac{N(\wp)-1}{3}}(\lambda-1)\frac{J\left(\omega_\wp^{\frac{N(\wp)-1}{6}}, \omega_\wp^{\frac23(N(\wp)-1)}\right)}{J\left(\omega_\wp^{\frac{N(\wp)-1}{3}}, \phi\right)}$$

is a character since

$$\frac{J\left(\omega_\wp^{\frac{N(\wp)-1}{6}}, \omega_\wp^{\frac{2}{3}(N(\wp)-1)}\right)}{J\left(\omega_\wp^{\frac{N(\wp)-1}{3}}, \phi\right)} = \frac{\mathfrak{g}\left(\omega_\wp^{\frac{N(\wp)-1}{6}}\right)\mathfrak{g}\left(\omega_\wp^{\frac{2}{3}(N(\wp)-1)}\right)}{\mathfrak{g}\left(\omega_\wp^{\frac{N(\wp)-1}{3}}\right)\mathfrak{g}(\phi)} = \omega_\wp^{-\frac{N(\wp)-1}{3}}(2),$$

which is the finite field version of the value $\tau_5'/\tau_1$.

*Exercise* 3.2. Verify the above using the multiplication formula $\mathfrak{g}(\chi)\mathfrak{g}(\phi\chi) = \mathfrak{g}(\chi^2)\mathfrak{g}(\phi)\overline{\chi}(4)$.

3.4. **Weil's theorem on Jacobi-sums.** Let $a, b \in \mathbb{Q}$ such that $a+b, a-b \notin \mathbb{Z}$. Let $M = lcd(a, b)$ and $K = \mathbb{Q}(\zeta_M)$ as above.

**Theorem 3.7** (Weil [27])**.** *The function from the set of all fractional ideals of $\mathbb{Z}[\zeta_M]$ coprime to $M$ to $\mathbb{C}^\times$ defined by the following formula is a Hecke (or Grössencharacter, see* [27] *or Definition 5.3 of* [11]*).*

$$\mathfrak{J}_{a,b}(\wp) = -J(\omega_\wp^{(q-1)a}, \omega_\wp^{(q-1)b}),$$

*where $q = |\mathbb{Z}[\zeta_M]/\wp|$, $\omega_\wp$ is chosen so that* (3) *is satisfied. In terms of Galois representation, this means for any fixed prime $\ell$, there exists a 1-dimensional Galois representation of $\chi_{a,b,\ell}$ of $G(M) \to \overline{\mathbb{Q}}_\ell$ which is unramified outside of primes dividing $M$ such that at any prime $\wp$ coprime to $M$,*

(8) $$\chi_{a,b,\ell}(Frob_\wp) = -J(\omega_\wp^{(q-1)a}, \omega_\wp^{(q-1)b}).$$

*Example* 3.4. For example, the Hekce character in Example 2.3 (of Part 2: Section 2.8) is the Grössencharacter in above theorem with $a = \frac{1}{4}, b = \frac{1}{2}$. More precisely, for each prime ideal $\mathfrak{p}$ of $\mathbb{Z}[i]$ coprime to 4, let $q$ be the norm of $\mathfrak{p}$ and $\psi_\mathfrak{p}$ be the order-4 multiplicative character such that $\psi_\mathfrak{p}(x) \equiv x^{\frac{q-1}{4}} \mod \mathfrak{p}$ for each $x \in \mathbb{Z}[\sqrt{-1}]$. The map that assigns $-\sum_{x \mod \mathfrak{p}} \psi_\mathfrak{p}(x)\psi_\mathfrak{p}^2(1-x) = -J(\psi_\mathfrak{p}, \psi_\mathfrak{p}^2)$ to $\mathfrak{p}$ extends to a Hecke (or Grössencharacter) character $\psi$ of $G_{\mathbb{Q}(\sqrt{-1})} = G(4)$.

3.5. **Katz's theorem.** The connection between $\mathbb{P}(HD, \cdot)$ and Galois representation $\rho_{HD,\ell}$ is the following result of Katz.

**Theorem 3.8** (Katz [12, 13])**.** *Let $\ell$ be a prime. Given a primitive pair of multi-sets $\alpha = \{a_1, \cdots, a_n\}$, $\beta = \{1, b_2, \cdots, b_n\}$ with $M = lcd(\alpha \cup \beta)$, for any datum $HD = \{\alpha, \beta; \lambda\}$ with $\lambda \in \mathbb{Z}[\zeta_M, 1/M] \smallsetminus \{0\}$ the following hold.*

- *i). There exists an $\ell$-adic Galois representation $\rho_{HD,\ell} : G(M) \to GL(W_\lambda)$ unramified almost everywhere such that at each prime ideal $\wp$ of $\mathbb{Z}[\zeta_M, 1/(M\ell\lambda)]$ with norm $N(\wp) = |\kappa_\wp|$,*

(9) $$Tr\rho_{HD,\ell}(Frob_\wp) = (-1)^{n-1}\omega_\wp^{(N(\wp)-1)a_1}(-1)\mathbb{P}(\alpha, \beta; 1/\lambda; \kappa_\wp),$$

  *where $Frob_\wp$ stands for the geometric Frobenius conjugacy class of $G(M)$ at $\wp$.*
- *iia). When $\lambda \neq 0, 1$, the dimension $d := dim_{\overline{\mathbb{Q}}_\ell} W_\lambda$ equals $n$ and all roots of the characteristic polynomial of $\rho_{DH,\ell}(Frob_\wp)$ are algebraic integers and have the same absolute value $N(\wp)^{(n-1)/2}$ under all archimedean embeddings.*
- *iib). When $\lambda \neq 0, 1$ and $HD$ is self-dual, then $W_\lambda$ admits a symmetric (resp. alternating) bilinear pairing if $n$ is odd (resp. even).*

*iii). When $\lambda=1$, dimension $d$ equals $n-1$. In this case if $HD$ is self-dual, then $\rho_{HD,\ell}$ has a subrepresentation $\rho_{HD,\ell}^{prim}$ of dimension $2\lfloor\frac{n-1}{2}\rfloor$ whose representation space admits a symmetric (resp. alternating) bilinear pairing if $n$ is odd (resp. even). All roots of the characteristic polynomial of $\rho_{HD,\ell}^{prim}(Frob_\wp)$ have absolute value $N(\wp)^{(n-1)/2}$, the same as (iia).*

Here and in what follows, when $\mathrm{ord}_\wp\lambda\geq 0$, the $\lambda$ in $\mathbb{P}(\alpha,\beta;\lambda;\kappa_\wp)$ is viewed as an element in the residue field $\kappa_\wp$.

**Remark 3.** *The sign $\omega_\wp^{(N(\wp)-1)a_1}(-1)$ as $\wp$ varies defines a character $\phi(M,a_1)$ of $G(M)$, which is trivial unless $ord_2\ M=-ord_2\ a_1=r\geq 1$. In the latter case it is the quadratic character corresponding to the Hilbert's quadratic norm residue symbol $\left(\frac{\zeta_{2^r}}{\cdot}\right)_2$ on the field $\mathbb{Q}(\zeta_M)$.*

3.6. **When the hypergeometric data are defined over $\mathbb{Q}$.** In this case $\rho_{HD,\ell}$ in the previous theorem is invariant under the twist by any $\tau\in G_\mathbb{Q}/G(M)$, so it can be lifted to a representation of $G_\mathbb{Q}$. As usual, the lifting is not unique, it is up to tensoring with any 1-dimensional character of the abelian group $G_\mathbb{Q}/G(M)\cong(\mathbb{Z}/M\mathbb{Z})^\times$. Due to [4, Theorem 1.5] by Beukers, Cohen and Mellit, one can choose the one particular lifting which corresponds to a submotive of $V_{\alpha,\beta}(\psi^{p_1+\cdots+p_s})$ (see Equation (22) of Part II).

**Theorem 3.9** (Katz, Beukers-Cohen-Mellit)**.** *Let $\alpha=\{a_1,\cdots,a_n\},\beta=\{1,b_2,\cdots,b_n\}$ be a primitive pair with $M=lcd(\alpha\cup\beta)$, and let $\lambda\in\mathbb{Z}[1/M]\smallsetminus\{0\}$. Suppose the hypergeometric datum $HD=\{\alpha,\beta;\lambda\}$ is defined over $\mathbb{Q}$. Assume that exactly $m$ elements in $\beta$ are in $\mathbb{Z}$. Then, for each prime $\ell$, there exists an $\ell$-adic representation $\rho_{HD,\ell}^{BCM}$ of $G_\mathbb{Q}$ with the following properties:*

*i). $\rho_{HD,\ell}^{BCM}|_{G(M)}\cong\rho_{HD,\ell}$.*

*ii). For any prime $p\nmid\ell\cdot M$ such that $ord_p\lambda=0$,*

$$Tr\,\rho_{HD,\ell}^{BCM}(Frob_p)=\phi(M,a_1)(Frob_p)\chi(\alpha,\beta;\mathbb{F}_p)H_p(\alpha,\beta;1/\lambda)\cdot p^{(n-m)/2}\in\mathbb{Z}. \tag{10}$$

*iii). When $\lambda=1$, $\rho_{HD,\ell}^{BCM}$ is $(n-1)$-dimensional and it has a subrepresentation, denoted by $\rho_{HD,\ell}^{BCM,prim}$, of dimension $2\lfloor\frac{n-1}{2}\rfloor$ whose representation space admits a symmetric (resp. alternating) bilinear pairing if $n$ is odd (resp. even). All roots of the characteristic polynomial of $\rho_{HD,\ell}^{BCM,prim}(Frob_p)$ have absolute value $p^{(n-1)/2}$.*

The character $\phi(M,a_1)$ in Remark 3 extends to a character of $G_\mathbb{Q}$. The character $\phi(M,a_1)$ in Theorem 3.9 refers to the extension with minimal conductor. When it is nontrivial, that is, when $\mathrm{ord}_2\ M=-\mathrm{ord}_2\ a_1=r\geq 1$, it has conductor $2^{r+1}$. In particular, when $\mathrm{ord}_2\ M=-\mathrm{ord}_2\ a_1=1$, $\phi(M,a_1)(\mathrm{Frob}_p)=\left(\frac{-1}{p}\right)$ is given by the Legendre symbol at odd primes $p$.

Given $\alpha=\{a_1,...,a_n\}$ and $\beta=\{b_1,...,b_n\}$ with $a_i,b_j\in\mathbb{Q}\cap[0,1)$ defined over $\mathbb{Q}$ and $M=lcd(\alpha\cup\beta)$, the following step function on the interval $[0,1)$ is introduced in [18] by Long:

$$e_{\alpha,\beta}(x):=\sum_{i=1}^n-\lfloor a_i-x\rfloor-\lfloor x+b_i\rfloor\,.$$

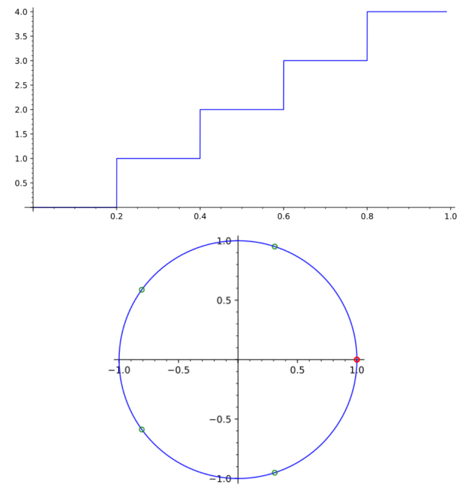
The value of $e_{\alpha,\beta}(x)$ jumps up (resp. down) only at $a_i$ (resp. $1-b_j$). When $p$ is an odd prime not dividing $M$, $0\leq k<p-1$ is an integer and the pair $\alpha,\beta$ is defined over $\mathbb{Q}$, $e_{\alpha,\beta}(\frac{k}{p-1})$ gives the collective exponent of $p$ in the $k$th summand of $H_p(\alpha,\beta;\lambda)$

$$H_q(\alpha,\beta;\lambda):=\frac{1}{1-q}\sum_{k=0}^{q-2}\prod_{j=1}^n\frac{\mathfrak{g}(\omega^{k+(q-1)a_j})\mathfrak{g}(\omega^{-k-(q-1)b_j})}{\mathfrak{g}(\omega^{(q-1)a_j})\mathfrak{g}(\omega^{-(q-1)b_j})}\,\omega^k\big((-1)^n\lambda\big). \tag{11}$$

Moreover when the pair $\alpha, \beta$ is defined over $\mathbb{Q}$, the shape of the graph is independent of $p$ as long as it is coprime to $lcd(\alpha, \beta)$.

We will compare it with the plot consisting of $\{e^{2\pi i a_j}\}_{j=1}^n$ and $\{e^{2\pi i b_j}\}_{j=1}^n$ on the unit circle mentioned in Lecture I. Both graphs encode the same information, while the step function represents multiplicity more clearly.

*Example* 3.5. For $\alpha = \{1/5, 2/5, 3/5, 4/5\}, \beta = \{0, 0, 0, 0\}$



The weight $w(HD)$ of a datum $HD = \{\alpha, \beta; \lambda\}$ is defined as

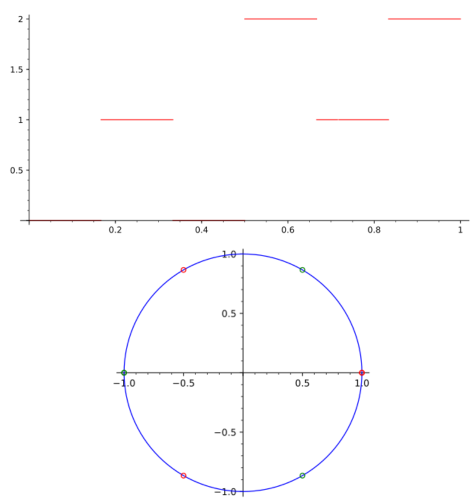
$$w(HD) = w(\alpha, \beta) := \max e_{\alpha,\beta}(x) - \min e_{\alpha,\beta}(x). \tag{12}$$

In order to compare what can be computed by `Magma`'s hypergeometric motive package, we further introduce the adjustment factor

$$t := -\min\{e_{\alpha,\beta}(x) \mid 0 \leq x < 1\} - \frac{n-m}{2}, \tag{13}$$

where $n = |\alpha|, m = \#\{b_j \mid b_j \in \mathbb{Z}\}$.

*Example* 3.6. $\alpha = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}\}$ and $\beta = \{0, 0, \frac{1}{3}, \frac{2}{3}\}$



In this case $max = 2, min = 0, max - min = 2, n = 4, m = 2, t = -1$.

For $HD$ defined over $\mathbb{Q}$, there is an efficient `Magma` program called "Hypergeometric Motives over $\mathbb{Q}$" implemented by Watkins (see [26]) which computes the characteristic polynomial of $\rho^{BCM}_{\{\alpha,\beta;\lambda\},\ell}[t]$ (resp. $\rho^{BCM,prim}_{\{\alpha,\beta;\lambda\},\ell}[t]$) at $\mathrm{Fr}_p$, the inverse of $\mathrm{Frob}_p$, for $p \nmid M\ell$ efficiently when $\lambda \neq 0,1$ (resp. $\lambda = 1$), where $t$ as in (13). Here $\rho[t]$ denotes the weight-$t$ Tate twist of a representation $\rho$ of $G_{\mathbb{Q}}$.

*Example* 3.7. `H:=HypergeometricData([1/5,2/5,3/5,4/5],[1,1,1,1]);`
$[w=4, t=0]$
`Factorization(EulerFactor(H,1,7));`
The output is $< 343 * \$.1^2 - 6 * \$.1 + 1, 1 >$

`Factorization(EulerFactor(H,-1,7));`
The output is $< 117649 * \$.1^4 + 8575 * \$.1^3 + 350 * \$.1^2 + 25 * \$.1 + 1, 1 >$, where $117649 = 7^6$.

`H2:=HypergeometricData([1/2,1/2,1/6,5/6],[0,0,1/3,2/3]);`
$[w=2, t=-1]$

`Factorization(EulerFactor(H2,1,5));`
The output is $< 5 * \$.1^2 + 2 * \$.1 + 1, 1 >$

`Factorization(EulerFactor(H2,-1,5));`
The output is
$< 5 * \$.1^2 - 4 * \$.1 + 1, 1 >$,
$< 5 * \$.1^2 + 2 * \$.1 + 1, 1 >$

3.7. **Modularity results.** Note that Theorem 3.9 implies that one can study a whole category of explicitly computatable Galois representations. They can be used to test standard conjectures or to discover new ones. For instance, according to Langlands general philosophies, these Galois representations are automorphic. In the remaining discussion, we focus on degree-2 irreducible subrepresentations constructed from hypergeometric data defined over $\mathbb{Q}$.

**Question 2.** *What do we get degree-2 irreducible subrepresentations of $G(M)$ or $G_{\mathbb{Q}}$ from a hypergeometric datum $HD = \{\alpha, \beta; \lambda\}$?*

In view of Theorem 3.9, here are some candidates.

- When $|\alpha| = |\beta| = 2$, $\lambda \neq 0, 1$.
- When $|\alpha| = |\beta| = 3$, self-dual and $\lambda = 1$; or when $HD$ has a CM background in view of Clausen formula (Section 1.10.1 (28)). See Proposition 3 of [16] for some modularity results.
- When $|\alpha| = |\beta| = 4$, self-dual and $\lambda = 1$. See [20, Theorem 2], [16, Theorem 4] for some modularity results.
- Other constructions from hypergeometric formulas, including 7 cases in which $|\alpha| = |\beta| = 6$, $\lambda = 1$ based on Whipple's formula (Section 1.10.6 (39)).

**Remark 4.** *When $\alpha = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, \beta = \{1, 1, 1, 1\}$, $\lambda = -1$, see* [21] *by McCarthy and Pananikolas for a result of how the 4-dimensional Galois representation is related to a Siegel modular form.*

**Question 3.** *How to construct degree-3 irreducible subrepresentations of $G_{\mathbb{Q}}$ from primitive hypergeometric data $HD = \{\alpha, \beta; \lambda\}$?*

A natural source of such representations comes from $|\alpha| = |\beta| = 3$ and $\lambda \neq 0, 1$. By Clausen formula (Theorem 2.17), they are symmetric squares of 2-dimensional representations of $G(M)$ when $\alpha, \beta$ are self-dual.

To obtain modularity result for degree-2 Galois representations of $G_{\mathbb{Q}}$, we use Theorems 3.1 and 3.2 recalled earlier.

3.7.1. *For* $HD = \{\alpha = \{r_1, r_2, 1-r_1, 1-r_2\}, \beta = \{1,1,1,1\}; \lambda = 1\}$. When $\alpha$ is defined over $\mathbb{Q}$, note that by Theorems 3.8 and 3.9, $\rho_{HD,\ell}^{BCM}$ of $G_{\mathbb{Q}}$ is 3-dimensional which decomposes into a direct sum of 2 subrepresentations $\rho_{HD,\ell}^{BCM,prim} \oplus \rho_{HD,\ell}^{BCM,1}$. Among them $\rho_{HD,\ell}^{BCM,prim}$ is 2-dimensional and $\rho_{HD,\ell}^{BCM,1}$ is 1-dimensional.

In this way we know that $\rho_{HD,\ell}^{BCM,prim}$ is modular in each case and are able to identify the corresponding weight-4 normalized Hecke eigenforms $f_\alpha$. The other piece $\rho_{HD,\ell}^{BCM,1}$ is of the form $\chi_\alpha \varepsilon_\ell$ where $\chi_\alpha$ is a finite order character and $\varepsilon_\ell$ stands for the $\ell$-adic cyclotomic character. The identification of $\chi_\alpha$ boils down to computing the values of $H_p(\alpha, \beta; 1) - a_p(f_\alpha)$.

**Theorem 3.10** (Long, Tu, Yui and Zudilin)**.** *Let $p > 5$ be a prime and $\alpha$ and $\beta$ as above. Then the following equality holds*:

$$H_p(\boldsymbol{\alpha}, \boldsymbol{\beta}; 1) = a_p(f_{\boldsymbol{\alpha}}) + \chi_{\boldsymbol{\alpha}}(p) \cdot p,$$

*where $a_p(f_{\boldsymbol{\alpha}})$ is the $p$-th coefficient of the normalized Hecke eigenform and $\chi_{\boldsymbol{\alpha}}$ is a Dirichlet character of order at most 2, whose precise description is given in Table 1.*

Table 1. The Hecke eigenforms for rigid hypergeometric Calabi–Yau threefolds

| $(r_1, r_2)$ | $f_\alpha(\tau)$ | level | LMFDB label | $\chi_{\boldsymbol{\alpha}}$ |
|---|---|---|---|---|
| $(\frac{1}{2}, \frac{1}{2})$ | $\eta_2^4\eta_4^4$ | $8 = 2^3$ | `8.4.a.a` | $\chi_1$ |
| $(\frac{1}{2}, \frac{1}{3})$ | $\eta_6^{14}/(\eta_2^3\eta_{18}^3) - 3\eta_2^3\eta_6^2\eta_{18}^3$ | $36 = 2^2 \cdot 3^2$ | `36.4.a.a` | $\chi_3$ |
| $(\frac{1}{2}, \frac{1}{4})$ | $\eta_4^{16}/(\eta_2^4\eta_8^4)$ | $16 = 2^4$ | `16.4.a.a` | $\chi_2$ |
| $(\frac{1}{2}, \frac{1}{6})$ | $\eta_4^3\eta_6^6\eta_{18}^2/(\eta_{12}^2\eta_{36}) - 3\eta_2^2\eta_6^6\eta_{36}^3/(\eta_4\eta_{12}^2)$ $+ 8\eta_2^3\eta_{12}^6\eta_{36}^2/(\eta_6^2\eta_{18}) - 16\eta_{12}^{12}/\eta_6^4$ | $72 = 2^3 \cdot 3^2$ | `72.4.a.b` | $\chi_1$ |
| $(\frac{1}{3}, \frac{1}{3})$ | $\eta_1^3\eta_3^4\eta_9 - 27\eta_3\eta_9^4\eta_{27}^3$ | $27 = 3^3$ | `27.4.a.a` | $\chi_1$ |
| $(\frac{1}{3}, \frac{1}{4})$ | $\eta_3^8$ | $9 = 3^2$ | `9.4.a.a` | $\chi_6$ |
| $(\frac{1}{3}, \frac{1}{6})$ | $\eta_6^{10}/\eta_{18}^2 - 27\eta_{18}^{10}/\eta_6^2 + 9\eta_6^7\eta_{54}^3/\eta_{18}^2 - 9\eta_2^3\eta_{18}^7/\eta_6^2$ | $108 = 2^2 \cdot 3^3$ | `108.4.a.a` | $\chi_3$ |
| $(\frac{1}{4}, \frac{1}{4})$ | $\eta_4^{10}/\eta_8^2 - 8\eta_8^{10}/\eta_4^2$ | $32 = 2^5$ | `32.4.a.a` | $\chi_1$ |
| $(\frac{1}{4}, \frac{1}{6})$ | $\eta_{12}^{32}/(\eta_6^{12}\eta_{24}^{12}) + 16\eta_6^4\eta_{24}^4$ | $144 = 2^4 \cdot 3^2$ | `144.4.a.f` | $\chi_2$ |
| $(\frac{1}{6}, \frac{1}{6})$ | | $216 = 2^3 \cdot 3^3$ | `216.4.a.c` | $\chi_1$ |
| $(\frac{1}{5}, \frac{2}{5})$ | $\eta_5^{10}/(\eta_1\eta_{25}) + 5\eta_1^2\eta_5^4\eta_{25}^2$ | $25 = 5^2$ | `25.4.a.b` | $\chi_5$ |
| $(\frac{1}{8}, \frac{3}{8})$ | | $128 = 2^7$ | `128.4.a.b` | $\chi_2$ |
| $(\frac{1}{10}, \frac{3}{10})$ | | $200 = 2^3 \cdot 5^2$ | `200.4.a.f` | $\chi_1$ |
| $(\frac{1}{12}, \frac{5}{12})$ | | $864 = 2^5 \cdot 3^3$ | `864.4.a.a` | $\chi_1$ |

Below we discuss two other classical formulas in light of Galois representations.

3.8. **Clausen formula.**

$${}_2F_1\left[\begin{matrix} c-s-\frac{1}{2} & s \\ & c \end{matrix}; \lambda\right]^2 = {}_3F_2\left[\begin{matrix} 2c-2s-1 & 2s & c-\frac{1}{2} \\ & 2c-1 & c \end{matrix}; \lambda\right].$$

This is an Orr type formula which expresses a product of two hypergeometric function in terms of another, see [25, §2.5]. As mentioned in Section 1, we can check the symmetric square of the Riemann scheme for the ${}_2F_1$ hypergeometric function on the left is the same as the Riemann scheme of the ${}_3F_2$ on the right. In [9], Evans and Greene showed the following formula.

**Theorem 3.11** (Evans-Greene)**.** *Let $C, S \in \widehat{\mathbb{F}_q^\times}$. Assume that $C \neq \phi$, and $S^2 \notin \{\varepsilon, C, C^2\}$. Then for $\lambda \neq 1$,*

$$
{}_2\mathbb{F}_1 \begin{bmatrix} C\overline{S}\phi & S \\ & C \end{bmatrix}; \lambda \Bigg]^2 = {}_3\mathbb{F}_2 \begin{bmatrix} C^2\overline{S}^2 & S^2 & C\phi \\ & C^2 & C \end{bmatrix}; \lambda \Bigg]
$$

$$
+ \phi(1-\lambda)\overline{C}(\lambda) \left( \frac{J(\overline{S}^2, C^2)}{J(\overline{C}, \phi)} + \delta(C)(q-1) \right).
$$

In terms of the Galois representation, this formula says the tensor product of the degree-2 Galois representation corresponds to $\{c - s - \frac{1}{2}\}, \{1, c\}$, which is 4-dimensional, which decomposes into its symmetric square, whose trace function is the ${}_3\mathbb{F}_2$ on the right plus its alternating square, whose trace function is $\phi(1-\lambda)\overline{C}(\lambda)\frac{J(\overline{S}^2, C^2)}{J(\overline{C}, \phi)}$ when $C \neq \varepsilon$. In this formula, the field field version closely resembles the classical one.

Next is a formula of Whipple.

$$
(14) \quad {}_7F_6 \begin{bmatrix} a & 1+\frac{a}{2} & c & d & e & f & g \\ & \frac{a}{2} & 1+a-c & 1+a-d & 1+a-e & 1+a-f & 1+a-g \end{bmatrix}; 1 \Bigg]
$$

$$
= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-e-f)\Gamma(1+a-e-g)}
$$

$$
\cdot {}_4F_3 \begin{bmatrix} a & e & f & g \\ & e+f+g-a & 1+a-c & 1+a-d \end{bmatrix}; 1 \Bigg],
$$

when both hand sides terminate.

Note that

- The parameter set of ${}_7F_6(1)$ is not primitive, i.e. $1 + \frac{a}{2}$ and $\frac{a}{2}$ are differed by 1
- The ${}_7F_6(1)$ is well-posed, meaning the upper and lower parameters sum to $1 + a$ in each column. Namely $\alpha = \{a_1, \cdots, a_7\}$, $\beta = \{1 + a - a_i, i = 1, \cdots, 7\}$.

Now we consider how to get self-dual parameter sets after cancelling $1 + \frac{a}{2}$ and $\frac{a}{2}$ out of the left hand side of this Whipple's formula.

- $\alpha$ being self-dual, "means" if $a \in \alpha$, $1 - a$ is also in $\alpha$. For this reason we assume

$$
c + d = 1, \quad f + g = 1.
$$

- $\beta = \{1 + a - a_i\}$ being self-dual requires $a = \frac{1}{2}$ and consequently $e = \frac{1}{2}$.

Thus we let

$$
a = \frac{1}{2}, \quad c + d = 1, \quad f + g = 1, \quad \text{and } e = \frac{1}{2}\left(-\frac{p}{2}\right)
$$

The choice of $e$ is to make it into a negative integer to guarantee both hand sides terminate while being $p$-adically close to $\frac{1}{2}$.

$$
{}_7F_6 \begin{bmatrix} \frac{1}{2} & \frac{5}{4} & c & 1-c & \frac{1-p}{2} & f & 1-f \\ & \frac{1}{4} & \frac{3}{2}-c & \frac{1}{2}+c & 1+\frac{p}{2} & \frac{3}{2}-f & \frac{1}{2}+f \end{bmatrix}; 1 \Bigg] =
$$

$$
\frac{\Gamma(\frac{p}{2})\Gamma(\frac{3}{2}-f)\Gamma(\frac{1}{2}+f)\Gamma(\frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(1+\frac{p}{2}-f)\Gamma(\frac{p}{2}+f)} \times \left( p \cdot {}_4F_3 \begin{bmatrix} \frac{1}{2} & \frac{1-p}{2} & f & 1-f \\ & 1-\frac{p}{2} & \frac{3}{2}-c & \frac{1}{2}+c \end{bmatrix}; 1 \Bigg] \right).
$$

Consequently, the hypergeometric datum corresponds to the left hand side is

$$HD_1(c,f) :=$$
$$\left\{\alpha_6(c,f) := \left\{\frac{1}{2}, c, 1-c, \frac{1}{2}, f, 1-f\right\}, \beta_6(c,f) = \left\{1, \frac{3}{2}-c, \frac{1}{2}+c, 1, \frac{3}{2}-f, \frac{1}{2}+f\right\}; 1\right\};$$

while the hypergeometric datum corresponds to the right hand side is

$$HD_2(c,f) := \left\{\alpha_4(f) := \left\{\frac{1}{2}, \frac{1}{2}, f, 1-f\right\}, \beta_4(c) := \left\{1, 1, \frac{3}{2}-c, \frac{1}{2}+c\right\}; 1\right\}.$$

We now get to Whipple's formula in terms of Galois representations

For $(c,f) \in \mathbb{Q}^2$ such that $HD_1, HD_2$ both primitive, we let $M(c,f) := lcd(HD_2)$, and $N(c,f) := lcd(\frac{1+2f-2c}{4}, \frac{3-2f-2c}{4})$ (which is due to an application of the Clausen formula, for details, see [16]). Either $N(c,f) = 2M(c,f)$ or $N(c,f) = M(c,f)$. The corresponding Galois groups either the same, or $G(N)$ is an index-2 subgroup of $G(M)$.

From Theorem 3.8, we know $\rho_{HD_1(c,f),\ell}$ is a $6-1=5$-dimensional representation of $G(M)$ which decomposes into a 4-dimensional primitive part, plus another 1-dimensional linear part. Similarly, $\rho_{HD_2(c,f),\ell}$ is a 3-dimensional Galois representation of $G(M)$ which decomposes into a 2-dimensional primitive part plus a 1-dimensional linear part.

**Theorem 3.12** (Li, Long, Tu [16])**.** *Assume $(c,f) \in \mathbb{Q}^2$ such that $HD_1, HD_2$ both primitive. Given any prime $\ell$,*

$$\rho_{HD_1(c,f),\ell}|_{G(N(c,f))} \cong (\epsilon_\ell \otimes \rho_{HD_2(c,f),\ell})|_{G(N(c,f))} \oplus \sigma_{sym,\ell}$$

*where $\epsilon_\ell$ is the $\ell$-adic cyclotomic character, and $\sigma_{sym,\ell}$ is a 2-dimensional representation of $G(N)$ that can be computed explicitly.*

Unlike the Clausen formula, the finite field analogue of Whipple is not as parallel as the classical formula, whose right hand side only corresponds to the trace function of $(\epsilon_\ell \otimes \rho_{HD_2(c,f),\ell})|_{G(N(c,f))}$. The other component corresponds to $\sigma_{sym,\ell}$ is usually highly nontrivial.

In [16], the authors further considered for what choices of $(c,f)$ the corresponding $HD_1$ is both primitive and defined over $\mathbb{Q}$. There are 7 such (un-ordered) cases. In each case, the authors described $\rho^{BCM}_{HD_1(c,f),\ell}(\mathrm{Frob}_p)$ by expressing its trace function in terms of 2 modular forms and a linear character. The information is listed in the next theorem.

**Theorem 3.13** (Li, Long, and Tu)**.** *For each pair $(c,f)$ in the list, $\rho^{BCM}_{HD_1(c,f),\ell}$ is modular (using the LMFDB label).*

| $(c,f)$ | $Tr\,\rho^{BCM}_{HD_1(c,f),\ell}(Frob_p)$ |
|---|---|
| $(\frac{1}{2},\frac{1}{2})$ | $a_p(f_{8.6.a.a}) + p\cdot a_p(f_{8.4.a.a}) + \left(\frac{-1}{p}\right)p^2$ |
| $(\frac{1}{2},\frac{1}{3})$ | $a_p(f_{4.6.a.a}) + p\cdot a_p(f_{12.4.a.a}) + \left(\frac{3}{p}\right)p^2$ |
| $(\frac{1}{3},\frac{1}{3})$ | $a_p(f_{6.6.a.a}) + p\cdot a_p(f_{18.4.a.a}) + \left(\frac{-1}{p}\right)p^2$ |
| $(\frac{1}{2},\frac{1}{6})$ | $p\cdot a_p(f_{8.4.a.a}) + p\cdot a_p(f_{24.4.a.a}) + \left(\frac{3}{p}\right)p^2$ |
| $(\frac{1}{6},\frac{1}{6})$ | $p^2\cdot a_p(f_{24.2.a.a}) + p^2\cdot a_p(f_{72.2.a.a}) + \left(\frac{-1}{p}\right)p^2$ |
| $(\frac{1}{5},\frac{2}{5})$ | $p\cdot a_p(f_{10.4.a.a}) + p\cdot a_p(f_{50.4.a.d}) + \left(\frac{-5}{p}\right)p^2$ |
| $(\frac{1}{10},\frac{3}{10})$ | $p^2\cdot a_p(f_{40.2.a.a}) + p^2\cdot a_p(f_{200.2.a.b}) + \left(\frac{-5}{p}\right)p^2$ |

**Remark 5.** *1. In the second column, the first modular form corresponds to an extension of $\sigma_{sym,\ell}$ to $G_{\mathbb{Q}}$, while the second modular form corresponds to the extension of $\rho^{BCM,prim}_{HD_2(c,f),\ell}$ to $G_{\mathbb{Q}}$.*

*2. In the proof of this formula, the character sum formula stated in Lemma 2.14 of Section 2 plays an important role.*

*Exercise* 3.3. For each pair $(c, f)$,

1). Compute $M(c, f)$ and $N(c, f)$, when will they be the same?

2). Draw the step function for each of the $HD_1(c, f)$, compute the corresponding weight function and compare it with the weight of the modular forms listed in the table.

3). If we swap $c$ and $f$, what happens to $HD_1(c, f)$ and $HD_2(c, f)$?

When $(c, f) = (\frac{1}{2}, \frac{1}{2})$, $H_p(HD_1) = a_p(f_{8.6.a.a}) + p \cdot a_p(f_{8.4.a.a}) + \left(\frac{-1}{p}\right) p^2$. It was first was conjectured by Koike [14] and was shown by Frechette-Ono-Papanikolas in [10].

Mortenson conjectured that for each odd prime $p$

$$ {}_6F_5 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 & 1 & 1 \end{bmatrix}; 1 \Bigg]_{p-1} \overset{?}{\equiv} a_p(f_{8.6.a.a}) \mod p^5. $$

The modulo $p^3$ claim was proved by Osburn, Straub, and Zudilin in [22].

Corresponds to Whipple formula's left hand side, numerically it is found that

$$ {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 & 1 & 1 & 1 \end{matrix}; 1\right]_{p-1} \overset{?}{\equiv} p \cdot a_p(f_{8.4.a.a}) \mod p^4. $$

The corresponding complex versions at infinity are

**Theorem 3.14.** *[Li, Long, Tu, [16]]*

$$ {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 & 1 & 1 \end{matrix}; 1\right] = 16 \int_{1/2+i/2}^{-1/2+i/2} \tau^2 f_{8.6.a.a}\left(\frac{\tau}{2}\right) d\tau $$

$$ {}_7F_6 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 & 1 & 1 & 1 \end{matrix}; 1\right] = \frac{32i}{\pi} \int_{1/2+i/2}^{-1/2+i/2} \tau f_{8.4.a.a}(\frac{\tau}{2}) d\tau, $$

*where the path is the hyperbolic geodesic from $\frac{1+i}{2}$ to $\frac{-1+i}{2}$, clockwise, which corresponds to an explicit circle in term of an modular function.*

The proof of Theorem 3.14 was based on the following idea of Zagier in [29]. Observe that

$$ {}_4F_3 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 \end{matrix}; 1\right] = \frac{1}{2\pi i} \oint_{|t|=1} {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix}; t\right] {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix}; 1/t\right] \frac{dt}{t}. $$

Letting $t$ to be the modular lambda function, Zagier obtained that

$$ {}_4F_3 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 \end{matrix}; 1\right] = \frac{16}{\pi^2} L(f_{8.4.a.a}, 2). $$

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