HYPERGEOMETRIC FUNCTIONS, CHARACTER SUMS AND APPLICATIONS

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ABSTRACT. We summarize several aspects of hypergeometric functions based on our recent work [14, 13, 16, 27, 30, 31, 32, 33] and our understanding of the subjects.

4. p-ADIC ASPECTS OF HYPERGEOMETRIC FUNCTIONS

4.1. p-adic Gamma functions.

Definition 1. The Mortita p-adic Gamma function $\Gamma_p(x)$ is defined for $n \in \mathbb{N}$ by

$$\Gamma_p(n) := (-1)^n \prod_{\substack{0<j<n \\ p\nmid j}} j,$$

and extends to $x \in \mathbb{Z}_p$ by defining $\Gamma_p(0) := 1$, and for $x \neq 0$,

$$\Gamma_p(x) := \lim_{n \to x} \Gamma_p(n),$$

where n runs through any sequence of positive integers p-adically approaching x.

For details of the following properties, see [32] by Long and Ramakrishna.

Theorem 4.1. *The function $\Gamma_p(\cdot)$ satisfies the following properties.*

1). $\Gamma_p(0) = 1$

2). $\dfrac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } |x|_p = 1 \\ -1, & \text{if } |x|_p < 1. \end{cases}$

3). $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}$ *where* $a_0(x) \in \{1, 2, \cdots, p\}$ *satisfies* $x - a_0(x) \equiv 0 \mod p$.

4). $\Gamma_p\left(\dfrac{1}{2}\right)^2 = (-1)^{\frac{p+1}{2}}$.

In particular, 3) above is the reflection formula for $\Gamma_p(\cdot)$. The following formula of Gross and Koblitz [20] provides a crucial link between Gauss sums and the p-adic Gamma function.

Theorem 4.2 (Gross and Koblitz). *For any integer* $0 \leq k < p-1$

$$\mathfrak{g}(\omega^{-k}) = -\pi_p^k \Gamma_p\left(\frac{k}{p-1}\right), \tag{1}$$

where ω is the Teichmuller character of $\mathbb{F}_p^\times$, π_p is a fixed root of $x^{p-1} + p = 0$ in $\mathbb{C}_p$, the additive character for the Gauss sum is $\zeta_p^{Tr_{\mathbb{F}_p}^{\mathbb{F}_q}(x)}$ of $\mathbb{F}_q$ where ζ_p is a primitive pth root of unity which is congruent to $1 + \pi_p$ modulo π_p^2.

Like the Gauss sums, the p-adic Gamma function also satisfies the multiplication formulas, see Theorem 11.6.14 of [11] by Cohen.

This note is based on Fang-Ting Tu's course on "Hypergeometric Functions" given at LSU in Fall 2020 and Ling Long's mini-lectures on "Hypergeometric Functions, Character Sums and Applications" given at University of Connecticut in 2021. Comments and suggestions will be appreciated. Special thanks to Dr. Bao Pham for his inputs.

Theorem 4.3. *Let p be a prime, N be a positive integer coprime to p. Then*

$$\prod_{0\le j<N} \Gamma_p\left(s+\frac{j}{N}\right) = c_{p,N}\frac{\Gamma_p(Ns)}{N^{[Ns-1-(Ns-1)\backslash p]}} \tag{2}$$

where

$$c_{p,N} = \begin{cases} \left(\frac{-p}{N}\right) & \text{if } N \text{ is odd} \\ (-1)^{N/2+1}\left(\frac{(-1)^{N/2+1}N/2}{p}\right)\Gamma_p(\frac{1}{2}) & \text{otherwise,} \end{cases}$$

where the notation $a\backslash p := (a-[a]_0)/p$ for $a \in \mathbb{Z}_p$ and $[a]_0$ is the first digit of a.

Proposition 4.4. *Let $m/n \in \mathbb{Q}$ where $m, n \in \mathbb{Z}$ such that $n \mid p-1$. Then $\Gamma_p(m/n)$ is an algebraic number in $\mathbb{Q}(\zeta_{np}, (-p)^{1/n})$.*

See Corollary 11.7.7 of [11].

Using Weil's Theorem (Theorem 3.7 of Part III), some combinations of Γ_p-values can be expressed explicitly in terms of Hecke characters.

Exercise 4.1. 1). Apply the Gross and Koblitz formula to Example 3.4 to show that for any prime $p \equiv 1 \mod 4$

$$\frac{\Gamma_p(1/4)^2}{\Gamma_p(1/2)} = a+bi, \quad \text{where } a,b\in\mathbb{Z}, p=a^2+b^2. \tag{3}$$

2). Similarly use $\mathfrak{J}_{\frac{1}{3},\frac{1}{3}}$ to derive a description for $\Gamma_p\left(\frac{1}{3}\right)^3$ when $p \equiv 1 \mod 3$.

Remark 1. *It is interesting to see these algebraic results are in contrast with Nesterenko's Theorem (Theorem 1.4 of Part I).*

Next we mention some analytic properties of $\Gamma_p(\cdot)$.

Theorem 4.5 (Morita, Barsky)**.** *For $a \in \mathbb{Z}_p$ the function $x \mapsto \Gamma_p(a+x)$ is locally analytic on $\mathbb{Z}_p$ and converges for $v_p(x) \ge \frac{1}{p} + \frac{1}{p-1}$.*

Definition 2. The *p-adic logarithm* is defined by

$$\log_p(1+x) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n},$$

which converges for $x \in \mathbb{C}_p$ with $|x|_p < 1$.

Set

$$G_k(a) = \Gamma_p^{(k)}(a)/\Gamma_p(a). \tag{4}$$

In particular, $G_0(a) = 1$.

Corollary 4.6. *For a in $\mathbb{Z}_p$, $G_1(a) = G_1(1-a)$, and $G_2(a) + G_2(1-a) = 2G_1^2(a)$.*

Proposition 4.7. *Let $x \in \mathbb{Z}_p^\times$. Than*

(1) $G_1(x+1) - G_1(x) = 1/x$.
(2) $G_2(x+1) - G_2(x) = G_1(x+1)^2 - G_1(x) - 1/x^2$.

See Proposition 2.2 of [24] by Kilbourn.

Corollary 4.8. *If* $1, 2, \cdots, k \in \mathbb{Z}_p^\times$*, then*

$$G_1(k+1) - G_1(1) = \sum_{j=1}^{k} \frac{1}{j},$$

the partial Harmonic sum, which is often denoted by H_k*. Similarly, if* $\frac{1}{2}, \frac{3}{2}, \cdots, k+\frac{1}{2} \in \mathbb{Z}_p^\times$*, then*

$$G_1\left(k+\frac{1}{2}\right) - G_1\left(\frac{1}{2}\right) = \sum_{j=1}^{k} \frac{1}{2j-1}.$$

Theorem 4.9 (Kazandzidis, [23])**.** *For* $0 \le m \le n$ *and prime* $p \ge 5$*,*

$$\binom{pn}{pm} \equiv \binom{n}{m} \mod p^3.$$

Theorem 4.10 (Robert and Zuber, [37])**.** *For any integer* $r \ge 1$

$$\binom{p^r n}{p^r m} \equiv \binom{p^{r-1} n}{p^{r-1} m} \mod p^{3r}. \tag{5}$$

Specializing $n = 2, m = 1$ into the above congruence implies

$$\frac{1}{p} \log_p 4^{p-1} + G_1\left(\frac{1}{2}\right) - G_1(0) = 0. \tag{6}$$

See [32] for a general machinery obtaining such relations using the Robert and Zuber congruences. Here is another one obtained in [32]

$$\frac{2}{p} \log_p(4^{p-1}) + G_1\left(\frac{1}{4}\right) + G_1\left(\frac{3}{4}\right) - G_1(1) - G_1\left(\frac{1}{2}\right) = 0. \tag{7}$$

Corollary 4.11. *For prime* $p > 2$ *and any integer* $r \ge 1$

$$2^{\frac{p^r - p^{r-1}}{2}} \equiv \left(\frac{2}{p}\right)\left[1 + \left(G_1\left(\frac{1}{2}\right) - G_1\left(\frac{1}{4}\right)\right)\frac{p^r}{2}\right] \mod p^{2r}.$$

It is known by Gauss that for prime $p \equiv 1 \mod 4$,

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \mod p, \quad p = a^2 + b^2, a \equiv 1 \mod 4.$$

Exercise 4.2. Use (3) to show that (7) is equivalent to the following super (namely stronger) congruence obtained by Chowla, Dwork and Evans in [10].

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \left(1 + \frac{2^{p-1}-1}{2}\right)\left(2a - \frac{p}{2a}\right) \mod p^2, \quad p = a^2 + b^2, a \equiv 1 \mod 4. \tag{8}$$

Next result is about the local analytic property of the Γ_p-function.

Theorem 4.12 ([32])**.** *For* $p \ge 5$*,* $r \in \mathbb{N}$*,* $a \in \mathbb{Z}_p, m \in \mathbb{C}_p$ *satisfying* $v_p(m) \ge 0$ *and* $t \in \{0, 1, 2\}$ *we have*

$$\frac{\Gamma_p(a + mp^r)}{\Gamma_p(a)} \equiv \sum_{k=0}^{t} \frac{G_k(a)}{k!}(mp^r)^k \mod p^{(t+1)r}.$$

The above result also holds for $t = 4$ *if* $p \ge 11$*.*

Definition 3. *Dwork's dash operation* — the map $\mathbb{Q} \cap \mathbb{Z}_p \to \mathbb{Q} \cap \mathbb{Z}_p$ defined by

$$r' = (r + [-r]_0)/p, \tag{9}$$

where $[a]_0$ is the first p-adic digit of a as before.

Despite the appearance, it has nothing to do with the usual derivative. By definition $1' = (1 + (p-1))/p = 1$.

Exercise 4.3. Let p be an odd prime, show that
1) $\left(\frac{1}{2}\right)' = \frac{1}{2}$,
2) for any integer $r \geq 1$, $\left(\frac{1-p^r}{2}\right)' = \frac{1-p^{r-1}}{2}$,
3) For a prime $p > 5$, give a formula for $(\frac{1}{5})'$.

Exercise 4.4. Let $\alpha = \{a_1, \cdots, a_n\}$ be defined over $\mathbb{Q}$ with $a_i \in (0,1)$ for each i. Show that for any $p \nmid lcd(\alpha)$, α as a set is closed under the p-adic Dwork dash operation.

The next Lemma is about how to convert Gamma quotients into p-adic Gamma quotients.

Lemma 4.13 ([32]). *Let $a \in (0,1] \cap \mathbb{Q}$.*

1) If $v_p(a) = 0$ then $\forall m, r \in \mathbb{N}$,

$$\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^{r-1})} = (-1)^m p^{mp^{r-1}} \frac{\Gamma_p(a + mp^r)}{\Gamma_p(a)} \frac{(a')_{mp^{r-1}}}{(a)_{mp^{r-1}}},$$

where a' as in Definition 3

2) Suppose $a + mp^r \in \mathbb{N}$ for each $r \in \mathbb{N}$. (Here, $a, m \in \mathbb{Q}$ but need **not** *be in $\mathbb{Z}$.) Then*

$$\frac{\Gamma(a + mp^r)}{\Gamma(a + mp^{r-1})} = (-1)^{a+mp^r} p^{a+mp^{r-1}-1} \Gamma_p(a + mp^r).$$

3) Let $a, b \in \mathbb{Q}$ and suppose $a - b \in \mathbb{Z}$ and $a, b \notin \mathbb{Z}_{\leq 0}$. If none of the numbers between a and b that differ from both by an integer are divisible by p then $\dfrac{\Gamma(a)}{\Gamma(b)} = (-1)^{a-b} \dfrac{\Gamma_p(a)}{\Gamma_p(b)}$. Equivalently, if $a \in \mathbb{Z}_p, n \in \mathbb{N}$ such that none of $a, a+1 \cdots, a+n-1$ is $p\mathbb{Z}_p$, then

$$(a)_n = (-1)^n \frac{\Gamma_p(a+n)}{\Gamma_p(a)}.$$

4.2. **Some origins of "supercongruences".**

4.2.1. *Beukers' conjecture.* Beukers studied the congruences satisfied by the Apéry numbers for the proofs of $\zeta(2), \zeta(3)$ being irrational in [7, 8]. One of the sequences is

$$u_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = {}_4F_3 \left[\begin{matrix} -n & -n & n+1 & n+1 \\ & 1 & 1 & 1 \end{matrix} ; 1 \right]$$

Beukers showed that for $m, r \geq 1, p > 3$,

$$u_{mp^r - 1} \equiv u_{mp^{r-1}-1} \mod p^{3r}.$$

He conjectured the following which was proved by Ahlgren and Ono.

Theorem 4.14 (Ahlgren, Ono [1]). *For each prime $p > 3$,*

$$u_{\frac{p-1}{2}} = {}_4F_3 \left[\begin{matrix} \frac{1-p}{2} & \frac{1-p}{2} & \frac{1+p}{2} & \frac{1+p}{2} \\ & 1 & 1 & 1 \end{matrix} ; 1 \right] \equiv a_p(\eta(2\tau)^4 \eta(4\tau)^4) \mod p^2,$$

where $\eta(\tau)$ is the Dedekind eta function.

Ahlgren and Ono [1] used Greene version of finite hypergeometric functions [19], their paper inspired Kilbourn's work [24].

4.2.2. *Ramanujan-type supercongruences.* Ramanujan-type gave a list of formulas for $1/\pi$, one of them is (see §1.9 of Part I)

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3}(6n+1)\frac{1}{4^n} = {}_4F_3\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{7}{6} \\ & 1 & 1 & \frac{1}{6} \end{matrix}; \frac{1}{4}\right].$$

Van Hamme made a list of Ramanujan-type supercongruence conjectures. E.g., for primes $p > 3$,

$${}_4F_3\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{7}{6} \\ & 1 & 1 & \frac{1}{6} \end{matrix}; \frac{1}{4}\right]_{p-1} \equiv \left(\frac{-1}{p}\right) p \mod p^4.$$

Z.W. Sun made more supercongruences conjectures based on his own computations [44]. Inspired by McCarthy and Osburn's paper [35] and Zudilin's work [48], Long proved it using a p-adic perturbation method applied to a formula of Gessel and Stanton [18]. This method was expanded in a paper by Long and Ramakrishna [32], adopted by Swisher in [45] to establish most of Van Hamme conjectures.

4.2.3. *Rodriguez-Villegas' conjectures.* Rodriguez-Villegas made a few supercongruences conjectures regarding hypergeometric Calabi-Yau manifolds. (The ${}_2F_1(1)$ and ${}_3F_2(1)$ cases are proved by Mortenson [36].) In particular,

Conjecture 1 (Rodriguez-Villegas)**.** *For each $\alpha = \{r_1, 1-r_1, r_2, 1-r_2\}$ such that $r_i \in (0,1)$ and α is defined over $\mathbb{Q}$, there exists a weight 4 Hecke cuspidal eigenform f_α satisfying that for all primes $p > 5$*

$${}_4F_3\left[\begin{matrix} r_1 & r_2 & 1-r_1 & 1-r_2 \\ & 1 & 1 & 1 \end{matrix}; 1\right]_{p-1} \equiv a_p(f_{\boldsymbol{\alpha}}) \mod p^3.$$

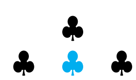
Kilbourn in [24], McCarthy in [34], and Fuselier-McCarthy in [17] each established one of these cases respectively. This conjecture was proved by Long, Tu, Yui and Zudilin for all 14 cases by two methods, to be explained later in this section.

In 2017, Roberts and Rodriguez-Villegas made a new supercongruence conjecture, we will recall in the end of this section.

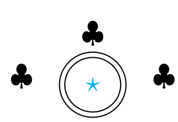
4.3. **The p-adic perturbation method.** Goal: to show that

$$\text{Truncated HGS } (\star) \equiv R \mod p^n.$$

(1) Understand the nature of R. Namely is it $\pm p^?, H_p, \gamma_p, a_p(f)$? ($R = \clubsuit?, \heartsuit?, \spadesuit?, \diamondsuit?$, say $R = \clubsuit$).
(2) Find a formula for R (put $\clubsuit$ into its background).



(3) Deform if necessary to peel off the desired truncated sum $(\star)$ as the major term and organize error terms in layers if possible



(4) Eliminate the error terms $\circledcirc$

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We will demonstrate how to use this method to prove this Van Hamme conjecture: for any prime $p > 2$

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1) \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 (-1)^k \equiv (-1)^{\frac{p-1}{2}} p \mod p^3. \tag{10}$$

It was first proved by Mortenson [36], another proof was by Zudilin using the Wilf-Zeilberger (WZ) method. It is a p-adic version of the following formula of Ramanujan.

$$\sum_{k=0}^{\infty} (4k+1) \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 (-1)^k = \frac{2}{\pi}.$$

Exercise 4.5. Show that

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1) \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 (-1)^k \equiv \sum_{k=0}^{p-1} (4k+1) \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 (-1)^k \mod p^3.$$

The right hand side of (10) is $(-1)^{\frac{p-1}{2}} p$; the left hand side is a truncated ${}_4F_3$ function which is well-posed. We first search for a well-posed ${}_4F_3(-1)$ series. Looking through the literature, we found an identity of Whipple [46, (5.1)] which says

$${}_4F_3 \left[\begin{matrix} a & 1+\frac{a}{2} & c & d \\ & \frac{a}{2} & 1+a-c & 1+a-d \end{matrix} ; -1 \right] = \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)},$$

if the left hand side terminates. Note that its parameter set has 3 variables, namely a, c, d. We next specify the values of a, c, d so that they are p-adically close to $\frac{1}{2}$. Letting $a = \frac{1}{2}, c = \frac{1}{2} + \frac{p}{2}, d = \frac{1}{2} - \frac{p}{2}$, the right hand side becomes $\frac{\Gamma(1-p/2)\Gamma(1+p/2)}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})} = (-1)^{\frac{p-1}{2}} p$, the desired right hand side. The left hand side is

$${}_4F_3 \left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1+p}{2} & \frac{1-p}{2} \\ & \frac{1}{4} & 1-\frac{p}{2} & 1+\frac{p}{2} \end{matrix} ; -1 \right] = \sum_{k=0}^{\frac{p-1}{2}} (1+4k) \frac{(\frac{1}{2})_k (\frac{1-p}{2})_k (\frac{1+p}{2})_k}{k! (1-\frac{p}{2})_k (1+\frac{p}{2})_k} (-1)^k,$$

as $\frac{1-p}{2}$ is a negative integer, the above series terminates. For k within the range of $[0, \frac{p-1}{2}]$, $\frac{(\frac{1}{2})_k (\frac{1-p}{2})_k (\frac{1+p}{2})_k}{k!(1-\frac{p}{2})_k (1+\frac{p}{2})_k}$ is p-adically integral and agrees with $\frac{(\frac{1}{2})_k^3}{k!^3} \mod p^2$.

Thus

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1) \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 (-1)^k \equiv \frac{\Gamma(1-\frac{p}{2})\Gamma(1+\frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} = (-1)^{\frac{p-1}{2}} p \mod p^2.$$

To achieve the congruence modulo p^3, we expand the terminating hypergeometric series (it terminates as $\frac{1-p}{2}$ is a negative integer) as an element in $\mathbb{Z}_p[[x]]$:

$${}_4F_3 \left[\begin{matrix} \frac{1-p}{2} & \frac{5}{4} & \frac{1-x}{2} & \frac{1+x}{2} \\ & \frac{1}{4} & 1+\frac{x}{2} & 1-\frac{x}{2} \end{matrix} ; -1 \right] = \sum_{k=0}^{\frac{p-1}{2}} (4k+1) \left(\frac{(\frac{1}{2})_k}{k!} \right)^3 (-1)^k + A_2 x^2 + A_3 x^4 + \cdots, \tag{11}$$

as a function of x, it is even. The goal next is to show $p \mid A_2$.

Though we can not obtain the left hand side of (11) directly from specializing parameters in the above Whipple's ${}_4F_3(-1)$ formula, we could use another formula of Whipple (see [5, pp. 28]).

$$
{}_6F_5\left[\begin{matrix} a & 1+\frac{a}{2} & b & c & d & e \\ & \frac{a}{2} & 1+a-b & 1+a-c & 1+a-d & 1+a-e \end{matrix} ; -1\right]
$$
$$
= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}\, {}_3F_2\left[\begin{matrix} 1+a-b-c & d & e \\ & 1+a-b, & 1+a-c \end{matrix} ; 1\right]
$$

Letting $a=\frac{1}{2}, b=\frac{1-x}{2}, c=\frac{1+x}{2}, e=\frac{1-p}{2}, d=1$, we have

$$
(12)\quad {}_6F_5\left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1-x}{2} & \frac{1+x}{2} & \frac{1-p}{2} & 1 \\ & \frac{1}{4} & 1+\frac{x}{2} & 1-\frac{x}{2} & \frac{1}{2} & 1+\frac{p}{2} \end{matrix} ; -1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(1+\frac{p}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{p}{2})}\, {}_3F_2\left[\begin{matrix} \frac{1}{2} & 1 & \frac{1}{2}-\frac{p}{2} \\ & 1+\frac{x}{2} & 1-\frac{x}{2} \end{matrix} ; 1\right]
$$

Since $\dfrac{\Gamma(\frac{1}{2})\Gamma(1+\frac{p}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{p}{2})} = p$, every x-coefficient of the above is in $p\mathbb{Z}_p$. Moreover, modulo p the left hand side of (11) is congruent to that of (12). So when we expand the left hand side of (11) in terms of x, the coefficients are all in $p\mathbb{Z}_p$. In particular, $p \mid A_2$ and this concludes the proof of (10).

Using this idea of perturbing classical formulas p-adically, Long and Ramakrishna were able to prove a variety of supercongruence results using existing classical hypergeometric formulas. This method works well if the right hand sides can be written as p-adic Gamma values (such as p-power or CM periods).

Example 4.1. Long and Ramakrishna showed in [32] the following conjectured by Kibelbek using an evaluation formula of Gessel and Stanton in [18]. For any prime $p \equiv 1 \mod 4$

$$
{}_3F_2\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix} ; -\frac{1}{8}\right]_{p-1} \equiv -\left(\frac{-2}{p}\right)\Gamma_p\left(\frac{1}{4}\right)^4 \mod p^3.
$$

The method has been adopted in [14] by Deines, Fuselier, Long, Swisher and Tu and in [45] by Swisher.

Exercise 4.6. In [32], it is shown that for any prime $p \equiv 1 \mod 6$

$$
{}_3F_2\left[\begin{matrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ & 1 & 1 \end{matrix} ; 1\right]_{p-1} \equiv \Gamma_p(\frac{1}{3})^6 \mod p^3.
$$

Meanwhile, in [14], it is pointed out that for $p \equiv 1 \mod 6$

$$
{}_3F_2\left[\begin{matrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ & 1 & 1 \end{matrix} ; 1\right]_{p-1} \overset{?}{\equiv} -\Gamma_p(\frac{1}{3})^3 \mod p^3.
$$

In the current case, $\Gamma_p(\frac{1}{3})^3$ is an algebraic number. So the right hand sides above have different absolute values. What could possibly be an explanation for the discrepancy between these two congruences? [An explanation was given in [14].]

There are a lot of recent developments on supercongruences, such as [6] by Barman and Saikia and there are q-version of supercongruences, see [22] by Guo and Zudilin and [21] by Guo and Schlosser for more discussions. In §4.10, we will introduce a residue sum method which may be used to create desired identities when no classical formulas are available for immediate use.

4.4. **Dwork's result.** Dwork [15] laid down a framework for p-adic hypergeometric functions. Here we only discuss his basic relevant results.

We use $\lfloor x \rfloor$ for the floor function of $x \in \mathbb{R}$ and denote $\{x\} := x - \lfloor x \rfloor$ the fractional part. For the discussion in this section, we work over the ring of p-adic integers $\mathbb{Z}_p$ with $p > 5$ being any fixed prime. Furthermore, for $r \in \mathbb{Z}_p$, let $[r]_0$ denote its first p-adic digit.

Lemma 4.15. *Given an integer k, $0 \le k < p$, and $r \in \mathbb{Z}_p^\times$, the rising factorial $(r)_k$ is in $\mathbb{Z}_p^\times$ if and only if $k \le [-r]_0$.*

Exercise 4.7. Prove the above Lemma.

Here is a fundamental congruence result proved by Dwork.

Theorem 4.16 (Dwork [15])**.** *Let p be a fixed prime, $\alpha = \{a_1, \cdots, a_n\}$ with $a_i \in \mathbb{Q}$, $\beta = \{1, \cdots, 1\}$ be two primitive multi-sets of the same length. Assume $p \nmid lcd(\alpha)$ and denote $\{a_1', \cdots, a_n'\}$ by α'. Let $F_m(\alpha, \beta; x) := \sum_{k=0}^m A(k) x^k$ which is a polynomial in $\mathbb{Z}_p[x]$ where $A(k) = \frac{\prod_{i=1}^n (a_i)_k}{k!^n}$ then for any positive integers s, t, m satisfying $t \ge s$*

$$F_{mp^t-1}(\alpha,\beta;x) F_{mp^{s-1}-1}(\alpha',\beta;x^p) \equiv F_{mp^{t-1}-1}(\alpha',\beta;x^p) F_{mp^s-1}(\alpha,\beta;x) \mod p^s \mathbb{Z}_p[x]. \tag{13}$$

Near $x \in \mathbb{Z}_p$ such that $F_{p-1}(\alpha, \beta; x) \ne 0 \mod p$, which is called the ordinary case, the quotient

$$\gamma_{\alpha,p}(x) := \lim_{s \to \infty} F_{mp^s-1}(\alpha,\beta;x) / F_{mp^{s-1}-1}(\alpha',\beta;x^p) \tag{14}$$

is a p-adic uniformally convergent function, referred to as the *Dwork unit root function*. See [47] by Yu and [9] by Beukers and Vlasenko for some later developments.

4.5. **From character sums to truncated hypergeometric functions.** In Lecture III, we recalled how to use a Magma package implemented by Watkins to evaluate hypergeometric character sums when the hypergeometric datum is defined over $\mathbb{Q}$. Recall also

$$H_q(\alpha,\beta;\lambda;\omega) := \frac{1}{1-q} \sum_{k=0}^{q-2} \prod_{j=1}^{n} \frac{\mathfrak{g}(\omega^{k+(q-1)a_j}) \mathfrak{g}(\omega^{-k-(q-1)b_j})}{\mathfrak{g}(\omega^{(q-1)a_j}) \mathfrak{g}(\omega^{-(q-1)b_j})} \, \omega^k\big((-1)^n \lambda\big). \tag{15}$$

Theorem 4.17 (Long, Tu, Yui and Zudilin)**.** *Let $HD = \{\alpha = \{r_1, r_2, 1-r_1, 1-r_2\}, \beta = \{1,1,1,1\}; \lambda = 1\}$, where $r_1, r_2 \in \{\frac12, \frac13, \frac14, \frac16\}$ or $(r_1, r_2) = (\frac15, \frac25)$, $(\frac{1}{10}, \frac{3}{10})$, $(\frac18, \frac38)$, $(\frac{1}{12}, \frac{5}{12})$. Let $p > 5$ be a prime. Then :*

$$H_p(\boldsymbol{\alpha}, \boldsymbol{\beta}; 1) = a_p(f_{\boldsymbol{\alpha}}) + \chi_{\boldsymbol{\alpha}}(p) \cdot p,$$

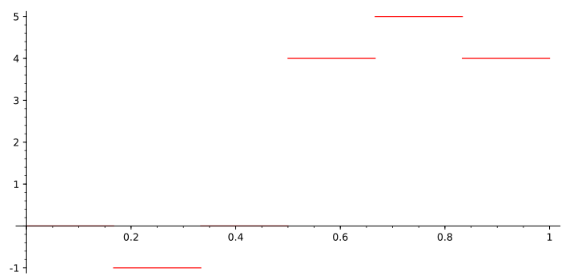
where $f_{\boldsymbol{\alpha}}$ has weight-4.

Applying the Gross-Koblitz formula to (15), one can obtain congruences between truncated hypergeometric series to the H_p-values.

Corollary 4.18. *Notation as above,*

$${}_4F_3 \begin{bmatrix} r_1 & r_2 & 1-r_1 & 1-r_2 \\ & 1 & 1 & 1 \end{bmatrix}; 1 \Bigg]_{p-1} \equiv H_p(\boldsymbol{\alpha}, \boldsymbol{\beta}; 1) \equiv a_p(f_{\boldsymbol{\alpha}}) \mod p.$$

When $\beta \ne (1, \cdots, 1)$, the situation gets more involved. In [31], Long used the step function in the previous section to obtain some generic results. For example, for $\alpha = \{\frac12, \frac12, \frac12, \frac12, \frac13, \frac23\}, \beta = \{1, 1, 1, 1, \frac16, \frac56\}$, the step function is shown below. Its minimal value is -1.



In this case, the H_p-function is not integral, but pH_p is. For all primes $p > 5$ and $\lambda \in \mathbb{Z}_p$

$$p \cdot {}_6F_5 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & 1 & 1 & 1 & \frac{7}{6} & \frac{5}{6} \end{bmatrix} ; \lambda \Bigg]_{p-1} \equiv p \cdot H_p(\alpha, \beta; \lambda) \mod p.$$

Note that in the above, $\frac{7}{6}$ instead of $\frac{1}{6}$ is used in the above generic congruence. When $\lambda = 1$, it is shown in [27] that

$$pH_p(\alpha, \beta; 1) = a_p(f_{4.6.a.a}) + p \cdot a_p(f_{12.4.a.a}) + \left(\frac{3}{p}\right) p^2$$

The authors also found the following numeric observation

$$p \cdot {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & \frac{5}{6} & \frac{7}{6} & 1 & 1 & 1 \end{matrix} ; 1 \right]_{p-1} \overset{?}{\equiv} a_p(f_{4.6.a.a}) \mod p^5.$$

The archimedean analogue of these supercongruences is the following identity obtained in [27].

$$\frac{1}{\pi} {}_6F_5 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & \frac{5}{6} & \frac{7}{6} & 1 & 1 & 1 \end{matrix} ; 1 \right] = 6i \oint_{|t_3|=1} \left(\frac{1}{3} + \tau + \tau^2\right) \cdot (f_{4.6.a.a}(\tau/2) - 27 f_{4.6.a.a}(3\tau/2)) d\tau,$$

where

$$t_3(\tau) = 4 \left(\frac{1}{3\sqrt{3}} \frac{\eta^6(\tau)}{\eta^6(3\tau)} + 3\sqrt{3} \frac{\eta^6(3\tau)}{\eta^6(\tau)} \right)^{-2}.$$

4.5.1. *Another application of the p-adic perturbation method.* Typically Dwork congruence is sharp in general. However it may satisfy a higher power congruence which is referred to as a supercongruence at a special fiber with either additional symmetry or admitting special decomposition.

For example, it is known that for $p \equiv 1 \mod 4$

$$(16) \qquad {}_2F_1 \left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & 1 \end{matrix} ; -1 \right] \equiv \left(\frac{2}{p}\right) \Gamma_p(\frac{1}{2}) \Gamma_p(\frac{1}{4})^2 \mod p^2.$$

If we dig into the nature of this statement, we see that the left hand side is a "period" of the Legendre curve $y^2 = x(1-x)(1+x)$, which admits complex multiplication by $\mathbb{Q}(-1)$. The right hand side, by (3) it is the value of a Hecke character. From Dwork's Theorem 4.16, we expect it will be $\gamma_{\{\frac{1}{2},\frac{1}{2}\},p}(-1)$. To prove the claim (16) modulo p^2, using the p-adic perturbation method, one can try to find a classic formula which after deforming the parameters p-adic will yield the desired result. Usually there are more than one ways to proceed. (For example Whipple gave many evaluation formulas for well-posed series at ± 1 in [46].) Here if we deform Kummer's evaluation formula (Equation (38) in Section 1.10.5). After specializing $b = \frac{1-p}{2}$, $c = \frac{1+p}{2}$, the left hand side becomes

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}\frac{1-p}{2} & \frac{1+p}{2} \\ & 1+p\end{matrix};\ -1\right] \quad &:= \quad \sum_{k=0}^{(p-1)/2} \frac{\left(\frac{1-p}{2}\right)_k \left(\frac{1+p}{2}\right)_k}{k!(1+p)_k}(-1)^k \\
&\overset{Lemma 4.13\,(3)}{=} \quad \sum_{k=0}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{k!^2} \left(1 - G_1(1+k)p + G_1(1)p\right)(-1)^k \mod p^2 \\
&= \quad {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix};\ -1\right]_{p-1} + e\cdot p
\end{aligned}
$$

where $e = \sum_{k=0}^{(p-1)/2} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{k!^2} \left(-G_1(1+k) + G_1(1)\right))(-1)^k \mod p^2$.

If we look at the right hand side of Kummer's evaluation formula, it is straightforward to check using the conversion between Gamma quotients and p-dic Gamma quotients that the right hand side of Equation (38) in Section 1.10.5 equals

$$
\begin{aligned}
&2^{(1-p)/2}\frac{\Gamma_p(1+p)\Gamma_p(1/2)}{(\Gamma_p(3/4+p/4)\cdot\Gamma_p((3+3p)/4)} \\
&\qquad\overset{Cor. 4.11}{=} \left(\frac{2}{p}\right)\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})^2\left(1+\left[G_1(1)-\frac{1}{2}G_1(\frac{1}{4})-\frac{1}{2}G_1(\frac{1}{2})\right]p\right) \mod p^2.
\end{aligned}
$$

In order to claim their respective major terms agree modulo p^2, one could seek any additional built in symmetry. For instance if we switch the choices of the parameter to $b = \frac{1+p}{2}$, $c = \frac{1-p}{2}$, then the same analysis as above yields

$$
{}_2F_1\left[\begin{matrix}\frac{1-p}{2} & \frac{1+p}{2} \\ & 1+p\end{matrix};\ -1\right] \equiv {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix};\ -1\right]_{p-1} - e\cdot p \mod p^2.
$$

While the right hand side is congruent to

$$
\left(\frac{2}{p}\right)\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{4})^2\left(1-\left[G_1(1)-\frac{1}{2}G_1(\frac{1}{4})-\frac{1}{2}G_1(\frac{1}{2})\right]p\right) \mod p^2.
$$

Comparing both hand sides, we conclude (16) holds. Meanwhile this approach depends on the availability of known evaluation formulas. We will give a different approach in a later section.

In general, for truncated hypergeometric series related to CM elliptic curves, one expects higher congruence will hold. See [12] by Coster and Van Hammer for a general result.

Exercise 4.8. Let p be an odd prime. Let $n = \frac{p-1}{2}$. Prove that for any integer $0 \le k \le n$,

$$
\binom{n}{k}\binom{n+k}{k}(-1)^k \equiv \frac{\left(\frac{1}{2}\right)_k^2}{k!^2} \mod p^2. \tag{17}
$$

4.6. **Atkin and Swinnerton-Dyer congruences.** In [4], Atkin and Swinnerton-Dyer proved that for any non-singular elliptic curve $E: \quad y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Z}$, then for any fix local uniformizer t, such as $t = \frac{x}{y}$ see [40], of E near infinity such that the expansion of $\frac{dx}{y} = \sum_{k\ge 1}\frac{a_k}{k}t^k$ in terms of t with $a_k \in \mathbb{Z}_p$, then the coefficients a_k satisfy that for any $n, s \ge 1$

$$
a_{np^s} - \#(p - E/\mathbb{F}_p)a_{np^{s-1}} + pa_{np^{s-2}} \equiv 0 \mod p^s. \tag{18}
$$

Such a congruence is called an *Atkin and Swinnerton-Dyer (ASD) congruence*, which is a p-adic analogue of Hecke three-term recursion satisfied by Hecke eigenforms. Such kind of ASD congruences can be understood via the commutative formal group laws to be recalled below. In their seminal paper, Atkin and Swinnerton-Dyer discovered that weight-k coefficients of noncongruence cuspidal modular form satisfy such type of congruences with the power of p^s being a strong power of $p^{(k-1)s}$. A general "block" form of their numeric observation was proved by Scholl in [39]. In sequence of paper, [28, 3, 29, 2], the authors studied the finer 3-term ASD congruences satisfied by certain genuine noncongruence modular forms. As an application, Li and Long used ASD congruences to prove in part a folklore conjecture about the characteristic of genuine noncongruence modular forms [25]. See [26] by Li and Long for a survey paper on these topics.

4.7. **Commutative formal group laws.** The aim of section is to connect Dwork's result 4.16 to local zeta functions of algebraic varieties.

A 1-dimensional Commutative Formal Group Law (1-CFGL) over a commutative ring R is a formal power series over R of the form $G(u,v) = u + v +$ higher degree terms $\in R[[u,v]]$ that satisfying commutativity $G(u,v) = G(v,u)$ and associativity, namely $G(G(u,v),w) = G(u,G(v,w))$. Two typical examples of formal groups are $\hat{G}_a(u,v) = u+v$, the additive formal group law; and $\hat{G}_m(u,v) = u+v+uv$, the multiplicative formal group law. A 1-CFGL is determined by its logarithm which takes the form

$$\ell(\tau) = \sum_{n\geq 1} \frac{b_n}{n}\tau^n,$$

where $b_n \in R$ and $b_1 = 1$; namely, $G(u,v) = \ell^{-1}(\ell(u)+\ell(v))$.

Next we recall Theorem A.8 of [43] by Stienstra and Beukers.

Theorem 4.19. *Let p be a prime and R be a $\mathbb{Z}_p$-algebra of characteristic 0 equipped with an emdomorphism $\sigma : R \to R$ such that $\sigma(a) \equiv a^p \mod pR$ for all $a \in R$. Let $\ell(\tau) = \sum_{n\geq 1} n^{-1} b_n \tau^n$ with $b_n \in R, b_1 = 1$. The following statements are equivalent:*

(i) $\ell(\tau)$ is the logarithm of a 1-CFGL over R.
(ii) $\ell^{-1}(\ell(u)+\ell(v)) \in R[[u,v]]$.
(iii) There exists $s_i \in R$ for $i \in \mathbb{Z}_{>0}$ such that for all $m,r \geq 1$,

$$b_{mp^r} + s_1\sigma(b_{mp^{r-1}}) + ps_2\sigma^2(b_{mp^{p-2}}) + \cdots + p^{r-1}s_r\sigma^r(b_{mp}) \equiv 0 \mod p^r \tag{19}$$

(iv) If R is p-adically complete and $b_p \in \mathbb{R}^\times$, then there exists $\gamma_{p,s} \in R^\times$ such that for all $m,r \geq 1$

$$\gamma_{p,s} = b_{mp^r}/\sigma(b_{mp^{r-1}}) \mod p^r. \tag{20}$$

4.8. **Stienstra's formal groups construction.**

Theorem 4.20 (Stienstra, [41, Theorem 1])**.** *Let K be a noetherian ring which is flat over $\mathbb{Z}$. Let $F_1,\ldots,F_r$ be a regular sequence of homogeneous polynomial in $K[T_0,\ldots,T_N]$ and let X be the scheme of $\mathbb{P}^N_K$ defined by the ideal $(F_1,\ldots,F_r)$. Put $d_i = \deg F_i$ and $d = \sum_{i=1}^r d_i$. Assume that X is flat over K and $d_i \geq d-N \geq 1$ for all i. Let*

$$J = \{i = (i_0,\ldots,i_N) \in \mathbb{Z}^{N+1} \mid i_0,\ldots,i_N \geq 1, i_0+\cdots+i_N = d\}.$$

Then there is a formal group law for $H^{N-r}(X,\hat{G}_{m,\mathcal{O}_X})$ over K of dimension $n = \binom{d-1}{N}$ whose logarithm $\ell(\tau)$ is the n-tuple $(\ell_i(\tau))_{i\in J}$ of power series in $\tau = (\tau_i)_{i\in J}$ given by

$$\ell_i(\tau) = \sum_{m\geq 1}\sum_{j\in J} m^{-1}\beta_{m,i,j}\tau_j^m,$$

where $\beta_{m,i,j}$ is the coefficient of $T_0^{mj_0-i_0}\cdots T_N^{mj_N-i_N}$ in $(F_1\cdots F_r)^{m-1}$.

Here, given a formal group G, a sheaf $\mathcal{J}$ of K-algebras on X and $i \in \mathbb{Z}_{\geq 0}$, $H^i(X, G_{\mathcal{J}})$ is a so-called Artin–Mazur functor from nilpotent K-algebras to abelian groups; see [41, Section 2]. For the above, $G = \hat{G}_m$, the multiplicative formal group law and $\mathcal{O}_X$ is the structure sheaf of X.

Recall from [42, Chapter 3] that the zeta function of X over $\mathbb{F}_p$ is

$$Z(X/\mathbb{F}_p;T) = \prod_{N=0}^{6} P_N(T)^{(-1)^{N+1}},$$

where $P_N(T) = \det\big(1 - TF_p | H^N_{\text{cris}}(X) \otimes \mathbb{Q}\big)$ and F_p is the Frobenius endomorphism on the de Rham–Witt complex on X (see [42, § 3.4]).

Theorem 4.21 (Stienstra)**.** *Let X be a smooth projective variety over $\mathbb{F}_p$ such that $H^N(X, \hat{G}_{m,X})$ is a* 1-CFGL *over $\mathbb{Z}_p$ with formal logarithm*

$$\sum_{m\geq 1} m^{-1}\beta_m \tau^m.$$

Take

$$P_N(T) = \det\big(1 - TF_p | H^N_{\text{cris}}(X) \otimes \mathbb{Q}\big),$$

where F_p is the Frobenius endomorphism on the de Rham–Witt complex on X. Then

$$P_N(T) = 1 + a_1 T + pa_2 T + \cdots + p^{k-1}a_k T^k \quad \textit{with } a_1, \ldots, a_k \in \mathbb{Z},$$

where $k = \deg P_N(T)$ is the N-th Betti number of X, and for all integers $m, n \geq 1$,

$$\beta_{mp^n} + a_1\beta_{mp^{n-1}} + pa_2\beta_{mp^{n-2}} + \cdots + p^{k-1}a_k\beta_{mp^{n-k}} \equiv 0 \mod p^n. \tag{21}$$

Corollary 4.22. *Assumptions as above. If $\beta_p \neq 0 \mod p$, then $\gamma_{p,s}$ of* (20) *is a reciprocal root of $P_N(T)$, i.e. $P_N(1/\gamma_{p,s}) = 0$.*

Proof. By (20) for any $m, n \geq 1$,

$$\frac{\beta_{mp^n}}{\beta_{mp^{n-1}}} \equiv \gamma_{p,s} \mod p^n. \tag{22}$$

Comparing with (21), when $m = n = 1$, this implies that $\gamma_{p,s} = -a_1 \mod p$. When $m = 1$, $n = 2$, the congruence (21) translates after division by β_p into

$$\frac{\beta_{p^2}}{\beta_p} + a_1 + pa_2\frac{\beta_1}{\beta_p} \equiv \gamma_{p,s} + a_1 + \frac{pa_2}{\gamma_{p,s}} \equiv 0 \mod p^2,$$

where (22) was used, resulting in

$$\gamma_{p,s}^2 + a_1\gamma_{p,s} + pa_2 \equiv 0 \mod p^2.$$

Continuing this inductively we deduce, for any integer $n \geq 0$,

$$\gamma_{p,s}^k + a_1\gamma_{p,s}^{k-1} + pa_2\gamma_{p,s}^{k-2} + \cdots + p^{k-1}a_k \equiv 0 \mod p^{k+n}.$$

Thus, $P_N(1/\gamma_{p,s}) = 0$ as required. □

Remark 2. *Stienstra pointed out in* [42, Section 3] *that when X is not smooth but all singularities are rational singularities, then $\gamma_{p,s}$ is a reciprocal root of $P_N(T) = \det\big(1 - TF_p | H^N_{\text{cris}}(\hat{X}) \otimes \mathbb{Q}\big)$ where $\hat{X}$ is any smooth model of X.*

For example if we consider the Hesse family

$$f(\mathbf{x};\psi) = x_0^3 + x_1^3 + x_2^3 - 3\psi x_0x_1x_2 = 0 \tag{23}$$

where $\mathbf{x} = (x_0, x_1, x_2)$. In this case $N = 2, d = 3, r = 1$ so $n = 1$ and

$$J = \{(i_0, \cdots, i_2) \mid i_0, i_1, i_2 \geq 1, i_0 + i_1 + i_2 = 3\} = \{(1,1,1)\}.$$

The logarithm of the corresponding 1-CFGL is of the form $\sum_{m\geq 1} m^{-1} b_m(\psi)\tau^m$, where $b_m(\psi)$ is the coefficient of $(x_0x_1x_2)^{m-1}$ in $f(\mathbf{x};\psi)^m$, so

$$b_m(\psi) = \sum_{k=0}^{m-1} \binom{m-1}{k,k,k,m-1-3k} (-3\psi)^{m-1-3k} = \sum_{k=0}^{m-1} \frac{(m-3k)_{3k}}{k!^3}(-3\psi)^{m-1-3k}.$$

When $m = p^s$ where $p > 3$ is a prime, for k within that range of $[0, p^s/3]$,

$$b_{p^s}(\psi) = \binom{p^s-1}{k,k,k,p^s-1-3k} = \sum_{k=0}^{p^s-1} \frac{(p^s-3k)_{3k}}{k!^3}(-3\psi)^{p^s-1-3k}$$
$$= (-3)^{p^s-1} \sum_k \frac{(1-\frac{p^s}{3})_k(\frac{1}{3}-\frac{p^s}{3})_k(\frac{2}{3}-\frac{p^s}{3})_k}{k!^3}(\psi)^{-3k}.$$

Exercise 4.9. Verify the last equality. [Hint: multiplication formula for rising factorial and reflection formula for the Gamma function.]

When $s = 1$, letting $\lambda = \psi^{-3}$

$$b_p(\psi) \equiv \sum_{k=0}^{p-1} \frac{(\frac{1}{3})_k(\frac{2}{3})_k}{k!^2}\lambda^k \mod p.$$

When $\psi^{-3} \neq 1$, the elliptic curve defined by $H(\psi) : f(\mathbf{x};\psi) = 0$ is non-singular; when $\psi^{-3} = 1$, $H(\psi)$ only has double point singularities which are rational singularities. Theorem 4.21 and its corollary imply that $\gamma_{p,s}$ is a reciprocal root of $P_1(T) = 1 + [p - \#H(\psi)/\mathbb{F}_p]T + pT^2$, i.e. $\gamma_{p,s}^2 + [p - \#H(\psi)/\mathbb{F}_p]\gamma_{p,s} + p = 0$. Coming back to Dwork's result for $\alpha = \{\frac{1}{3}, \frac{2}{3}\}, \beta = \{1,1\}$, when p is ordinary for $H(\psi)$,

$$\gamma_{\{\frac{1}{3},\frac{2}{3}\},p}(\lambda) := \lim_{s\to\infty} {}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}; \lambda\right]_{p^s-1} / {}_2F_1\left[\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}; \lambda^p\right]_{p^{s-1}-1}$$

Using an argument of Beukers and Vlasenko, one can show that

$$\gamma_{\{\frac{1}{3},\frac{2}{3}\},p}(\lambda) = \gamma_{p,s}(\psi), \quad \text{where } \lambda = \psi^{-3}. \tag{24}$$

Thus Dwork's unit root is a reciprocal root of $P_1(T)$ as well. If $\psi \in \mathbb{Z}$, the Hesse pencil $H(\psi)$ is modular. Putting together we have the following result

Theorem 4.23. *Let f_ψ be the corresponding weight-2 cuspform, then the conclusion is equivalent to when p is a good ordinary prime, Dwork p-adic unit root $\gamma_{\{\frac{1}{3},\frac{2}{3}\},p}$ is a root of $T^2 - a_p(f_\psi)T + p = 0$.*

Exercise 4.10. Use the model

$$X_1 - Y_1 - Y_2 - Y_3 = 0, \quad 27\lambda X_1^3 = Y_1Y_2Y_3$$

(Example 2.4 in Section 2.8.2), apply Theorem 4.20 to obtain the formal logarithm and then compare it with the computation above.

Back to the 14 families of hypergeometric Calabi-Yau threefold family, the following Proposition follows from Stienstra's results.

Proposition 4.24. *For any of the 14* $\alpha = \{r_1, 1-r_1, r_2, 1-r_2\}$ *defined over* $\mathbb{Q}$, $\beta = \{1,1,1,1\}$ $\psi = \lambda = 1$, *if* p *is ordinary, then* $\gamma_{\alpha,p}(1)$ *defined by (14) is a root of* $T^2 - a_p(f_\alpha)T + p^3$. *In particular*

$$\gamma_{\alpha,p}(1) \equiv a_p(f_\alpha) \mod p^3. \tag{25}$$

4.9. **Approaching supercongruences from the perspective of Dwork unit roots.**

Lemma 4.25. *Let* $k \in \mathbb{Z}_{\geq 0}$, $a = [k]_0$ *and* $b = (k-a)/p$, *that is,* $k = a + bp$. *Then for any* $r \in \mathbb{Z}_p^\times$

$$\frac{(r)_k}{(1)_k} = \frac{-\Gamma_p(r+k)}{\Gamma_p(1+k)\,\Gamma_p(r)} \frac{(r')_b}{(1)_b} \cdot \big((r'+b)p\big)^{\nu(a,[-r]_0)},$$

where

$$\nu(a,x) = -\left\lfloor \frac{x-a}{p-1} \right\rfloor = \begin{cases} 0 & \text{if } a \leq x, \\ 1 & \text{if } x < a < p. \end{cases} \tag{26}$$

See [33, Lemma 2].

It follows one can peel off the contribution of the first digit of k from $\frac{(r)_k}{(1)_k}$. Assume $[k]_0 = a$, i.e. $k = a + bp$ where $a \in [0, p-1]$. (Using the notation introduced in Theorem 4.3, $b = k\backslash p$.) Then

$$\frac{(r)_{a+bp}}{(1)_{a+bp}} = \frac{(r)_a}{a!} \frac{(r')_b}{(1)_b} \left(1 + \frac{b}{r'}\right)^{\nu(a,[-r]_0)} \frac{\Gamma_p((r+a)+bp)\Gamma_p(1+a)}{\Gamma_p(r+a)\Gamma_p((1+a)+bp)}. \tag{27}$$

We now approach (16) in a different way. Let $p \equiv 1 \mod 4$ be a prime, $r = \frac{1}{2}$ and hence $r' = \frac{1}{2}$. We now know the right hand side is $\gamma_{\{\frac{1}{2},\frac{1}{2}\},p}(-1)$. Thus to prove (16), it is equivalent to show that

$$F(\alpha,\beta;-1)_{p^2-1}/F(\alpha,\beta;-1)_{p-1} \equiv F(\alpha,\beta;-1)_{p-1} \mod p^2. \tag{28}$$

$$\begin{aligned} F(\alpha,\beta;-1)_{p^2-1} &= \sum_{a,b=0}^{p-1} \frac{\left(\frac{1}{2}\right)_{a+bp}^2}{(1)_{a+bp}^2}(-1)^{a+bp} = \sum_{a,b=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_{a+bp}^2}{(1)_{a+bp}^2}(-1)^{a+bp} \\ &\overset{(27)}{\equiv} \sum_{a,b=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_a^2}{(1)_a^2} \frac{\left(\frac{1}{2}\right)_b^2}{(1)_b^2}(-1)^{a+b}[1 + 2(G_1(\frac{1}{2}+a) - G_1(1+a))p] \mod p^2 \end{aligned}$$

Thus (28) is equivalent to for each prime $p > 2$

$$\sum_{a=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_a^2}{(1)_a^2}(-1)^a[G_1(\frac{1}{2}+a) - G_1(1+a)] \equiv 0 \mod p.$$

Note that difference of two G_1 values may be written using the partial Harmonic sum, as in Corollary 4.8.

Exercise 4.11. Show that the above can be derived by the following identity: that for any positive integer n,

$$\sum_{k=0}^{2n} \binom{2n}{k}^2 (-1)^k \left(H_{2n-k} - H_k\right) = 0, \tag{29}$$

where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is the partial Harmonic sum. Such an identity can be checked numerically easily.

4.10. **Identities from the residue sum method.** Using the residue sum formula for rational functions, many identities can be obtained. We will illustrate by examples.

Consider the rational function

$$R(t) = \frac{\prod_{i=1}^{n}(t-i)^2}{\prod_{i=0}^{2n}(t+i)}.$$

Using partial fraction expansion,

$$R(t) = \sum_{k=0}^{n} \frac{A_k}{(t+k)},$$

where

$$A_k = R(t)(t+k)\big|_{t=-k} = \frac{\prod_{i=1}^{n}(k+i)^2}{k!(2n-k)!}(-1)^k = \frac{(k+n)!^2(-2n)_k}{k!^3(2n)!}.$$

The residue sum

$$\sum_{k=0}^{n} A_k = \sum_{k=0}^{n} Res_{t=-k}R(t) = -Res_{t=\infty}R(t) = 1, \tag{30}$$

as the degree of the numerator of $R(t)$ is by 1 less than the degree of its denominator.

It follows that

$${}_3F_2\left[\begin{matrix}-2n & n+1 & n+1 \\ & 1 & 1\end{matrix}\; ;\; 1\right] = \binom{2n}{n}. \tag{31}$$

This can be also derived from Pfaff-SaalSchütz evaluation.

If we change the rational function to

$$R(t) = \frac{\prod_{i=1}^{n}(t-i)^2}{\prod_{i=0}^{n}(t+i)^2}.$$

This calculation can be carried out in a similar way.

Write

$$R(t) = \sum_{k=0}^{n} \frac{B_k}{t+k} + \sum_{k=0}^{n} \frac{A_k}{(t+k)^2},$$

where

$$A_k = R(t)(t+i)^2|_{t=-k} = \frac{(k+1)_n^2}{k!^2(n-k)^2} = \binom{n}{k}^2\binom{n+k}{k}^2.$$

Exercise 4.12. Verify that

$$B_k = \frac{d\left(R(t)(t+k)^2\right)}{dt}|_{t=-k} = A_k\left(-2H_{n+k} - 2H_{n-k} + 4H_k\right).$$

For this choice of rational function, the residue sum is 0 as the denominator is degree 2 higher than the numerator. It follows

$$\sum_{k=0}^{n} \binom{n}{k}^2\binom{n+k}{k}^2 (H_{n+k} + H_{n-k} - 2H_k) = 0. \tag{32}$$

Exercise 4.13. Use the residue sum method to show that

$$\sum_{k=1}^{n} \binom{n+k}{k}^2\binom{n}{k}^2 (1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k) = 0. \tag{33}$$

The identity (33) was originally proved using the WZ method and was used in [1] by Ahlgren and Ono to prove Beukers's conjecture. It was also used in [24] by Kilbourn.

As seen above, the residue sum method can create lots of identities involving rising factorials by construction.

Example 4.2. Can the identity stated in Exercise 4.11 be proved using the residue sum technique?

When we use these identities to the p-adic setting, the following Lemma will be useful.

Lemma 4.26. *For any $t \in \mathbb{Z}_p$ and an integer $a \in \{0, 1, \ldots, p-1\}$, we have*

$$\frac{d}{dt}(t)_a = (t)_a\left(G_1(t+a) - G_1(t) + \frac{\nu(a, [-t]_0)}{t+[-t]_0}\right);$$

$$\frac{d^2}{dt^2}(t)_a = (t)_a\left(\left(G_1(t+a) - G_1(t) + \frac{\nu(a, [-t]_0)}{t+[-t]_0}\right)^2 + G_2(t+a) - G_2(t) - G_1(t+a)^2 + G_1(t)^2 - \frac{\nu(a, [-t]_0)}{(t+[-t]_0)^2}\right),$$

where $\nu(a, x)$ is defined in (26)*. Notice that $t + [-t]_0 = pt'$.*

See [33, Lemma 3].

4.11. **Supercongruences for rigid Calabi-Yau three-folds.** We now briefly outline the following result of [33].

Theorem 4.27 (Long, Tu, Yui and Zudilin)**.** *Let $HD = \{\alpha = \{r_1, r_2, 1-r_1, 1-r_2\}, \beta = \{1,1,1,1\}; \lambda = 1\}$, where $r_1, r_2 \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ or $(r_1, r_2) = (\frac{1}{5}, \frac{2}{5})$, $(\frac{1}{10}, \frac{3}{10})$, $(\frac{1}{8}, \frac{3}{8})$, $(\frac{1}{12}, \frac{5}{12})$. Let $p > 5$ be a prime. Then*

$${}_4F_3\begin{bmatrix} r_1 & r_2 & 1-r_1 & 1-r_2 \\ & 1 & 1 & 1 \end{bmatrix}; 1\Bigg]_{p-1} \equiv a_p(f_{\boldsymbol{\alpha}}) \mod p^3.$$

We will give two methods for proving this result using the p-adic perturbation method. Namely,

a). When p is ordinary, by (25) the right hand side can be replaced by $\gamma_{\alpha,p}(1)$, hence can be computed by Dwork's unit root formula.

b). Or $a_p(f_\alpha) = H_p(\alpha, \beta; p) - \chi_\alpha(p)p$, which can be expanded by the Gross-Kobliz formula.

Below we only outline how to approach using the first route. Interested readers are referred to [33] for details regarding both proofs.

Theorem 4.28 (Long, Tu, Yui and Zudilin)**.** *Let $\alpha = \{r_1, r_2, 1-r_1, 1-r_2\}$ be one of the fourteen multi-sets defined over $\mathbb{Q}$ and p a prime such that $r_1, r_2 \in \mathbb{Z}_p^\times$. Let $F_s(\alpha) := F(\alpha, \{1,1,1,1\}; 1)_{p^s-1}$. Then for any integer $s \geq 1$,*

$$F_{s+1}(\alpha) \equiv F_s(\alpha) F_1(\alpha) \mod p^3.$$

It is obtained in the following steps.

I) We re-organize the data as follows. Note that the dash operation preserves the multi-set; we numerate the entries in the multi-set $\{r_1, r_2, r_3, r_4\}$ in such a way that

$$r_1' \leq r_2' \leq r_3' \leq r_4'. \tag{34}$$

This inequality, the structure of the entries in $\{r_1, r_2, r_3, r_4\} = \{r_1', r_2', r_3', r_4'\}$ and the trivial property $(1-r)' = 1 - r'$ for rational $r \in (0,1) \cap \mathbb{Z}_p^\times$ result in

$$r_1 + r_4 = r_2 + r_3 = 1 \quad \text{and} \quad r_1' + r_4' = r_2' + r_3' = 1.$$

Furthermore, denote

$$a_j := [-r_j]_0 = pr_j' - r_j, \quad \text{for } j = 1, 2, 3, 4.$$

From the ordering chosen in (34), if $i < j$ then $a_i - a_j = p(r'_i - r'_j) - (r_i - r_j) \le -(r_i - r_j) < 1$, hence $a_i \le a_j$ as they are both integers. Putting together,

$$a_1 \le a_2 \le a_3 \le a_4 \quad \text{and} \quad a_1 + a_4 = a_2 + a_3 = p - 1.$$

The extra factors appearing in (27) are collected in the expression

$$\Lambda_{\boldsymbol{\alpha}}(a + bp) := \prod_{j=1}^{4} \left(1 + \frac{b}{r'_j}\right)^{\nu(a,[-r_j]_0)} = \begin{cases} 1 & \text{if } 0 \le a \le a_1, \\ (1 + b/r'_1) & \text{if } a_1 < a \le a_2, \\ (1 + b/r'_1)(1 + b/r'_2) & \text{if } a_2 < a \le a_3 \end{cases} \tag{35}$$

(we omit the other cases in view of their irrelevance) and, for $0 \le a < p$, the p-adic order of the Pochhammer quotient

$$\frac{\prod_{j=1}^4 (r_j)_a}{a!^4} \tag{36}$$

is equal to $s \in \{0, 1, 2, 3, 4\}$ if and only if $a_s < a \le a_{s+1}$, where we additionally set $a_0 = -1$ and $a_5 = p - 1$.

II) Apply the key reduction formula (27) one has

$$\begin{aligned} F_{s+1}(\alpha) &= \sum_{a=0}^{p-1} \sum_{b=0}^{p^s-1} \frac{\prod_{j=1}^4 (r_j)_{a+bp}}{(1)_{a+bp}^4} \\ &= \sum_{b=0}^{p^s-1} \frac{\prod_{j=1}^4 (r'_j)_b}{b!^4} \sum_{a=0}^{p-1} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4} \Lambda_\alpha(a + bp) \frac{\prod_{j=1}^4 \Gamma_p((r_j + a) + bp)}{\Gamma_p((1 + a) + bp)^4}. \end{aligned}$$

It follows from Theorem 4.12 that

$$\frac{\prod_{j=1}^4 \Gamma_p((r_j + a) + bp)}{\Gamma_p((1 + a) + bp)^4} \equiv \frac{\prod_{j=1}^4 \Gamma_p(r_j + a)}{\Gamma_p(1 + a)^4} \big(1 + J_1(a)\, bp + J_2(a)\, (bp)^2\big) \mod p^3, \tag{37}$$

where the coefficients $J_1(a)$ and $J_2(a)$ are given by

$$\begin{aligned} J_1(a) &= J_1(a, \boldsymbol{\alpha}) := \sum_{j=1}^4 \big(G_1(r_j + a) - G_1(1 + a)\big), \\ J_2(a) &= J_2(a, \boldsymbol{\alpha}) := 10 G_1(1 + a)^2 - 4 G_1(1 + a) \sum_{j=1}^4 G_1(r_j + a) \\ &\qquad + \sum_{1 \le j < \ell \le 4} G_1(r_j + a) G_1(r_\ell + a) + \frac{1}{2} \sum_{j=1}^4 \big(G_2(r_j + a) - G_2(1 + a)\big). \end{aligned} \tag{38}$$

Thus (27) and from (37),

$$\begin{aligned} F_{s+1}(\alpha) &= \sum_{a=0}^{p-1} \sum_{b=0}^{p^s-1} \frac{\prod_{j=1}^4 (r_j)_{a+bp}}{(1)_{a+bp}^4} \\ &\equiv \sum_{b=0}^{p^s-1} \frac{\prod_{j=1}^4 (r'_j)_b}{b!^4} \sum_{a=0}^{p-1} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4} \times \Lambda_\alpha(a + bp) \big(1 + J_1(a) \cdot bp + J_2(a) \cdot (bp)^2\big) \mod p^3. \end{aligned}$$

III) Comparing the right hand side, the claim will follow if one can establish the following claim For any $b \in \mathbb{Z}_{\geq 0}$, the congruence

$$\sum_{a=0}^{p-1} \frac{\prod_{j=1}^{4}(r_j)_a}{a!^4}\big(\Lambda(a+bp)\big(1+J_1(a)\cdot bp + J_2(a)\cdot (bp)^2\big)-1\big) \equiv 0 \mod p^3 \tag{39}$$

From the p-adic evaluation of (36) and the definition of $\Lambda(a+bp)$, we conclude that the left-hand side modulo p^3 is a quadratic polynomial $C_0 + C_1 b + C_2 b^2$ in b, with the constant term $C_0 = 0$, and

$$\begin{aligned} C_1 &= p\sum_{a=0}^{a_1} \frac{\prod_{j=1}^{4}(r_j)_a}{a!^4} J_1(a) + \sum_{a=a_1+1}^{a_2} \frac{\prod_{j=1}^{4}(r_j)_a}{a!^4}\left(\frac{1}{r_1'} + pJ_1(a)\right) \\ &\quad + \left(\frac{1}{r_1'} + \frac{1}{r_2'}\right) \sum_{a=a_2+1}^{a_3} \frac{\prod_{j=1}^{4}(r_j)_a}{a!^4}; \\ C_2 &= p^2 \sum_{a=0}^{a_1} \frac{\prod_{j=1}^{4}(r_j)_a}{a!^4} J_2(a) + \frac{p}{r_1'} \sum_{a=a_1+1}^{a_2} \frac{\prod_{j=1}^{4}(r_j)_a}{a!^4} J_1(a) \\ &\quad + \frac{1}{r_1' r_2'} \sum_{a=a_2+1}^{a_3} \frac{\prod_{j=1}^{4}(r_j)_a}{a!^4}, \end{aligned}$$

where the terms, which are zero modulo p^3 for trivial reasons, are discarded.

IV) The congruence (39) holds if the following stronger claims $C_1 \equiv 0 \mod p^3$ and $C_2 \equiv 0 \mod p^3$ hold. These two are discovered numerically first. They are shown to be true using identities created from the residue sums of

$$R(t) = \frac{\prod_{j=1}^{4}\prod_{i=1}^{a_j}(t-i+pr_j')}{\prod_{i=0}^{p-1}(t+i)^2}.$$

In this formula, there are extra pr_j' term are used to boost the congruence claims to be held from modulo p^2 to modulo p^3. See [33] for the most technical part of the proof.

4.12. **Roberts and Rodriguez-Villegas conjecture and new numeric supercongruence conjectures.**

Conjecture 2 (Roberts, Rodriguez-Villegas [38])**.** *Let $\alpha = \{a_1, \cdots, a_n\}$, $\beta = \{1, \cdots, 1\}$ be multisets satisfying defined over $\mathbb{Q}$ and $a_i \in (0,1)$, $\lambda = \pm 1$. Let A be the unique submotive of the hypergeometric motive corresponding to $\{\alpha, \beta; \lambda\}$ with hodge number $h^{0,n-1}(A) = 1$ and r the smallest positive integer such that $h^{r,n-1-r}(A) = 1$. For any $p \nmid lcd(\alpha, \beta)$ and ordinary for $\{\alpha, \beta; \lambda\}$, there is a p-adic unit $\mu_{\alpha,\beta;\lambda,p}$ depending on the hypergeometric datum such that for any integer $s \geq 1$*

$$F(\alpha,\beta;\lambda)_{p^s-1}/F(\alpha,\beta;\lambda)_{p^{s-1}-1} \equiv \mu_{\alpha,\beta;\lambda,p} \mod p^{rs}.$$

Question 1. *How to find such kinds of special hypergeometric data?*

One recent approach by Li, Long and Tu was to use a Whipple ${}_7F_6(1)$-formula and its finite field analogue, as mentioned in Lecture III. In this paper and also in [31], more numeric supercongruences are identified. We leave the proofs to interested readers.

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