HYPERGEOMETRIC FUNCTIONS, CHARACTER SUMS AND APPLICATIONS

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Abstract. We summarize several aspects of hypergeometric functions based on our recent work [8, 7, 14, 22, 23, 24, 25, 26] and our understanding of the subjects.

1. Hypergeometric functions over finite fields, 1st approach

In this section, we will approach hypergeometric functions over finite fields in a way parallel to the development recalled above. For details, see [14].

1.1. **Gauss sums and Jacobi sums.** To begin, fix an odd prime p and let $\mathbb{F}_q$ be a finite field of size q, where $q = p^e$. Recall that a *multiplicative character* χ on $\mathbb{F}_q^\times$ is a group homomorphism

$$\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times \text{ (or } \mathbb{C}_p^\times),$$

and the set $\widehat{\mathbb{F}_q^\times}$ of all multiplicative characters on $\mathbb{F}_q^\times$ forms a cyclic group of order $q-1$ under multiplication. Throughout, we fix the following notations

$$\varepsilon := \text{ the trivial character in } \widehat{\mathbb{F}_q^\times}$$
$$\phi := \text{ the quadratic character in } \widehat{\mathbb{F}_q^\times},$$

so that $\varepsilon(a) = 1$ for all $a \neq 0$, and ϕ is nontrivial such that $\phi^2 = \varepsilon$. Use $\overline{\chi}$ to denote the complex conjugate or the inverse of χ. We extend the definition of each character $\chi \in \widehat{\mathbb{F}_q^\times}$ to all of $\mathbb{F}_q$ by setting $\chi(0) = 0$, including $\varepsilon(0) = 0$. In some textbooks, such as [2] use a different convention that $\varepsilon(0) = 1$.

For δ for $\chi \in \widehat{\mathbb{F}_q^\times}$, or $x \in \mathbb{F}_q$, define

$$\delta(\chi) := \delta_\varepsilon(\chi) := \begin{cases} 1 & \text{if } \chi = \varepsilon, \\ 0 & \text{if } \chi \neq \varepsilon; \end{cases}$$

$$\delta(x) := \delta_0(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

This definition will allow us to describe formulas which hold for all characters, without having to separate cases involving trivial characters.

The orthogonal properties:

- $$\sum_{x \in \mathbb{F}_q} \Phi(x) = 0.$$

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- For $\chi, \varphi \in \widehat{\mathbb{F}_q^\times}$
$$\langle \chi, \varphi \rangle = \frac{1}{q-1} \sum_{x \in \mathbb{F}_q^\times} \chi(x)\overline{\varphi}(x) = \delta(\chi\overline{\varphi}).$$

Let $\zeta_N := e^{2\pi i/N}$. We fix a primitive p^{th} root of unity ζ_p and $A, B \in \widehat{\mathbb{F}_q^\times}$. We define the *Gauss sum* by

$$\mathfrak{g}(A) := \sum_{x \in \mathbb{F}_q^\times} A(x)\Phi(x), \quad \Phi(x) := \zeta_p^{\mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(x)}$$

where $\mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(x) := x + x^p + x^{p^2} + \cdots + x^{p^{e-1}}$ is the trace of x viewed as a surjective linear map from $\mathbb{F}_q$ to $\mathbb{F}_p$. The Gauss sum is the finite field analogue of the gamma function. It follows from the orthogonality of the additive character $\Phi(x)$ that

$$\mathfrak{g}(\varepsilon) = -1.$$

The Gauss sums depends on the choice of the additive character $\Phi(x)$, which depends on the primitive root ζ_p. Like the Gamma function, Gauss sums also satisfy reflection formula

$$\mathfrak{g}(A)\mathfrak{g}(\overline{A}) = qA(-1) - (q-1)\delta(A) \tag{1}$$

and the following multiplication formula. As a corollary, when $A \neq \varepsilon$,

$$|\mathfrak{g}(A)| = \sqrt{q}.$$

Theorem 1.1 (Hasse-Davenport Relation, see Theorem 11.3.5 of [2])**.** *Let $m \in \mathbb{N}$ and $q = p^e$ be a prime power with $q \equiv 1 \pmod m$. For any multiplicative character $\psi \in \widehat{\mathbb{F}_q^\times}$, we have*

$$\prod_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi^m = \varepsilon}} \mathfrak{g}(\chi\psi) = -\mathfrak{g}(\psi^m)\psi(m^{-m}) \prod_{\substack{\chi \in \widehat{\mathbb{F}_q^\times} \\ \chi^m = \varepsilon}} \mathfrak{g}(\chi).$$

In [34], Yamamoto proved a conjecture of Hasse which says the following:

Theorem 1.2 (Yamamoto)**.** *Let $M \geq 4$ be an even integer, $p \equiv 1 \pmod M$ be a prime, then the reflection formula and the multiplication formulas by the divisors n of M are the only two types of relations connecting the Gauss sums $\mathfrak{g}(\chi)$ for $\chi \in \widehat{\mathbb{F}_p^\times}$ satisfying $\chi^M = \varepsilon$, when considered as ideals in the ring of algebraic integers.*

Next we define the finite field analogue of the Beta function, the Jacobi sum, as follows. For $A, B \in \widehat{\mathbb{F}_q^\times}$

$$J(A, B) := \sum_{x \in \mathbb{F}_q} A(x)B(1-x).$$

In relation to Gauss sums

$$J(A, B) = \frac{\mathfrak{g}(A)\mathfrak{g}(B)}{\mathfrak{g}(AB)} + (q-1)B(-1)\delta(AB).$$

In particular,

$$|J(A, B)| = \begin{cases} \sqrt{q}, & \text{if } A, B, AB \neq \varepsilon, \\ q-2, & \text{if } A = B = \varepsilon, \\ 1, & \text{otherwise.} \end{cases}$$

We will use the following notation

$$\binom{A}{B} := -B(-1)J(A, \overline{B}).$$

1.2. **Lagrange inversion.** Below is a finite field analogue of the Lagrange inversion formula. We state the version where the basis of complex valued functions on the finite field is comprised of all multiplicative characters in $\widehat{\mathbb{F}_q^\times}$, together with $\delta(x)$.

Theorem 1.3 ([18] Theorem 2.7)**.** *Let p be an odd prime, $q = p^e$, and suppose $f : \mathbb{F}_q \to \mathbb{C}$ and $g : \mathbb{F}_q \to \mathbb{F}_q$ are functions. Then*

$$\sum_{\substack{y\in\mathbb{F}_q \\ g(y)=g(x)}} f(y) = \delta(g(x)) \sum_{\substack{y\in\mathbb{F}_q \\ g(y)=0}} f(y) + \sum_{\chi\in\widehat{\mathbb{F}_q^\times}} f_\chi \chi(g(x)),$$

where

$$f_\chi = \frac{1}{q-1} \sum_{y\in\mathbb{F}_q} f(y)\overline{\chi}(g(y)).$$

Compared with the classical formula (Day 1: Theorem 1.6), the assumptions $f(0) = 0$, $f'(0) \neq 0$, i.e. the map being one-to-one near 0, are not required. Greene pointed out that it is also the reason why the finite field version cannot be used to determine coefficients when f is not a one-to-one function.

From the Lagrange inversion, one can obtain a finite field version of the binomial theorem (Day 1: Equation (8)).

Lemma 1.4. *For any multiplicative character $A \in \widehat{\mathbb{F}_q^\times}$ and $x \in \mathbb{F}_q$, we have*

$$\overline{A}(1-x) = \delta(x) + \frac{1}{q-1} \sum_{\chi\in\widehat{\mathbb{F}_q^\times}} J(A\chi, \overline{\chi})\chi(-x) = \delta(x) + \frac{-1}{q-1} \sum_{\chi} \binom{A\chi}{\chi} \chi(x).$$

We will end this discussion by another Hasse-Davenport relation. Let $\mathbb{F}_q$ be a finite field, $\mathbb{F}_{q^r}$ be its degree-r extension of $\mathbb{F}_q$. Let $\mathrm{Tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^r}}$ and $\mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^r}}$ be the trace and norm maps from $\mathbb{F}_{q^r}$ to $\mathbb{F}_q$. Then for any $\chi \in \widehat{\mathbb{F}_q^\times}$, $\chi_r(x) = \chi(N_{\mathbb{F}_q}^{\mathbb{F}_{q^r}}(x))$ is a multiplicative character for $\mathbb{F}_{q^r}^\times$.

Theorem 1.5 (Hasse and Davenport)**.** *Notation as above. Let $\mathfrak{g}(\chi)$ be the Gauss sum of χ in $\mathbb{F}_q$, and $\mathfrak{g}(\chi_r)$ be the Gauss sum of χ_r in $\mathbb{F}_{q^r}$. Then,*

$$-\mathfrak{g}(\chi_r) = (-\mathfrak{g}(\chi))^r. \tag{2}$$

See [19] by Ireland-Rosen for a proof.

1.3. **A dictionary between the complex and finite field settings.** We now list a dictionary that we will use for convenience. Let $N \in \mathbb{N}$, and $a, b \in \mathbb{Q}$ with common denominator N.

$$\begin{array}{ccc}
\frac{1}{N} & \to & \text{an order } N \text{ character } \eta_N \in \widehat{\mathbb{F}_q^\times} \\
a = \frac{i}{N},\ b = \frac{j}{N} & \to & A, B \in \widehat{\mathbb{F}_q^\times},\ \ A = \eta_N^i,\ B = \eta_N^j \\
x^a & \to & A(x) \\
x^a x^b = x^{a+b} & \to & A(x)B(x) = AB(x) \\
a + b & \to & A \cdot B \\
-a & \to & \overline{A} \\
\Gamma(a) & \to & \mathfrak{g}(A) \\
(a)_n = \Gamma(a+n)/\Gamma(a) & \to & (A)_\chi = \mathfrak{g}(A\chi)/\mathfrak{g}(A) \\
B(a,b) & \to & J(A,B) \\
\int_0^1 \quad \mathrm{d}x & \to & \sum_{x \in \mathbb{F}} \\
\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi},\ \ a \notin \mathbb{Z} & \to & \mathfrak{g}(A)\mathfrak{g}(\overline{A}) = A(-1)q,\ \ A \neq \varepsilon \\
(ma)_{mn} = m^{mn} \displaystyle\prod_{i=1}^m \left(a + \frac{i}{m}\right)_n & \to & (A^m)_{\psi^m} = \psi(m^m) \displaystyle\prod_{i=1}^m (A\eta_m^i)_\psi
\end{array}$$

1.4. **Finite field period functions and normalized period functions.** We now introduce a finite field version of P- and F-functions parallel to the classical case. We start with a natural analogue to ${}_1P_0$ (Day 1: Equation(12)) by letting

$$(3) \qquad {}_1\mathbb{P}_0[A; \lambda; q] := \overline{A}(1 - \lambda),$$

for $A \in \widehat{\mathbb{F}_q^\times}$ and $\lambda \in \mathbb{F}_q$. We inductively define

$$(4) \quad {}_{n+1}\mathbb{P}_n \left[\begin{matrix} A_1 & A_2 & \dots & A_{n+1} \\ & B_2 & \dots & B_{n+1} \end{matrix}; \lambda; q\right] := \sum_{y \in \mathbb{F}_q} A_{n+1}(y)\overline{A}_{n+1}B_{n+1}(1-y) \cdot {}_n\mathbb{P}_{n-1} \left[\begin{matrix} A_1 & A_2 & \dots & A_n \\ & B_2 & \dots & B_n \end{matrix}; \lambda y; q\right],$$

which corresponds to (Day 1: Equation(15)) via the dictionary in §1.3. We note the asymmetry among the characters A_i (resp. B_j) in the definition. Part of the reason we start with the analogue of the period, rather than the hypergeometric function is because the periods are related to point counting. When there is no ambiguity in our choice of field $\mathbb{F}_q$, we will leave out the q in this notation and simply write

$${}_{n+1}\mathbb{P}_n \left[\begin{matrix} A_1 & A_2 & \dots & A_{n+1} \\ & B_2 & \dots & B_{n+1} \end{matrix}; \lambda\right] := {}_{n+1}\mathbb{P}_n \left[\begin{matrix} A_1 & A_2 & \dots & A_{n+1} \\ & B_2 & \dots & B_{n+1} \end{matrix}; \lambda; q\right].$$

When $n = 1$ we have,

$${}_2\mathbb{P}_1 \left[\begin{matrix} A & B \\ & C \end{matrix};\ \lambda\right] = \sum_{x \in \mathbb{F}_q} B(x)\overline{B}C(1-x)\overline{A}(1-\lambda x).$$

By Lemma 1.4, we can rewite ${}_2\mathbb{P}_1$ in terms of Jacobi sums as follows:

$$
{}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] = \begin{cases} J(B, C\overline{B}), & \text{if } \lambda = 0 \\ \dfrac{B(-1)}{q-1} \displaystyle\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} J(A\chi,\overline{\chi})J(B\chi,\overline{C\chi})\chi(\lambda), & \text{if } \lambda \neq 0 \end{cases}
$$

$$
= \begin{cases} J(B, C\overline{B}), & \text{if } \lambda = 0 \\ \dfrac{BC(-1)}{q-1} \displaystyle\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(\lambda), & \text{if } \lambda \neq 0. \end{cases}
$$

A general formula for the ${}_{n+1}\mathbb{P}_n$ period function highlighting this symmetry is:

$$
\begin{aligned}
&{}_{n+1}\mathbb{P}_n\left[\begin{matrix} A_1 & A_2 & \dots & A_{n+1} \\ & B_2 & \dots & B_{n+1} \end{matrix}; \lambda\right] \\
&\qquad = \frac{(-1)^{n+1}}{q-1}\cdot\left(\prod_{i=2}^{n+1} A_iB_i(-1)\right)\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{A_1\chi}{\chi}\binom{A_2\chi}{B_2\chi}\cdots\binom{A_{n+1}\chi}{B_n\chi}\chi(\lambda) \\
&\qquad\qquad + \delta(\lambda)\prod_{i=2}^{n+1} J(A_i, \overline{A_i}B_i).
\end{aligned}
$$

This is similar to Greene's definition [17, Def. 3.10], though we note that our binomial coefficient differs from Greene's by a factor of $-q$, and we define the value at $\lambda = 0$ differently.

In the classical case, a hypergeometric function is normalized to have constant term 1, obtained from the corresponding ${}_{n+1}P_n$ function divided by its value at 0. Here we similarly normalize the finite field period functions. Define

$$
{}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] = \frac{1}{J(B, C\overline{B})}\,{}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right]. \tag{5}
$$

The ${}_2\mathbb{F}_1$ function satisfies

1) ${}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,0\right] = 1;$
2) ${}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] = {}_2\mathbb{F}_1\left[\begin{matrix} B & A \\ & C \end{matrix}\,;\,\lambda\right]$, if $A,\ B \neq \varepsilon$, and $A,\ B \neq C$. Intuitively, with the additional Jacobi sum factor one can rewrite the right hand side of (5) using the finite field analogues of rising factorials with the roles of A and B being symmetric: for $\lambda \neq 0$,

$$
\begin{aligned}
{}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] &= \frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \frac{g(A\chi)g(\overline{\chi})}{g(A)}\frac{g(B\chi)g(\overline{C\chi})}{g(B)g(\overline{C})}\chi(\lambda) \\
&= \frac{1}{1-q}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} (A)_\chi(\varepsilon)_{\overline{\chi}}(B)_\chi(\overline{C})_{\overline{\chi}}\cdot\chi(\lambda).
\end{aligned}
$$

More generally, we define

$$
{}_{n+1}\mathbb{F}_n\left[\begin{matrix} A_1 & A_2 & \cdots & A_{n+1} \\ & B_2 & \cdots & B_{n+1} \end{matrix}\,;\,\lambda\right] := \frac{1}{\prod_{i=2}^{n+1} J(A_i, B_i\overline{A_i})}\,{}_{n+1}\mathbb{P}_n\left[\begin{matrix} A_1 & A_2 & \dots & A_{n+1} \\ & B_2 & \dots & B_{n+1} \end{matrix}; \lambda\right]. \tag{6}
$$

1.5. **Finite fields version of hypergeometric formulas.** Here we list some hypergoemetric formulas in the finite fields setting. These are finite field version of the formulas in Day 1: Subsection 1.10.

1.5.1. *Degree 1 transformations.*

Proposition 1.6 (Solutions around singularities)**.** *For any characters A, B, $C \in \widehat{\mathbb{F}_q^\times}$, and $\lambda \in \mathbb{F}_q$, we have*

$$
\begin{aligned}
{}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] &= ABC(-1)\overline{C}(\lambda)\ {}_2\mathbb{P}_1\left[\begin{matrix} \overline{C}B & \overline{C}A \\ & \overline{C} \end{matrix}\,;\,\lambda\right] + \delta(\lambda)J(B, C\overline{B}), \\
&= ABC(-1)\overline{A}(\lambda)\ {}_2\mathbb{P}_1\left[\begin{matrix} A & \overline{C}A \\ & \overline{B}A \end{matrix}\,;\,1/\lambda\right] + \delta(\lambda)J(B, C\overline{B}), \\
&= B(-1)\ {}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & AB\overline{C} \end{matrix}\,;\,1-\lambda\right].
\end{aligned}
$$

Proposition 1.7 (Euler and Pfaff)**.** *For any characters A, B, $C \in \widehat{\mathbb{F}_q^\times}$, and $\lambda \in \mathbb{F}_q$, we have*

$$
\begin{aligned}
{}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] &= \overline{A}(1-\lambda)\ {}_2\mathbb{P}_1\left[\begin{matrix} A & C\overline{B} \\ & C \end{matrix}\,;\,\frac{\lambda}{\lambda-1}\right] + \delta(1-\lambda)J(B, C\overline{AB}), \\
&= \overline{B}(1-\lambda)\ {}_2\mathbb{P}_1\left[\begin{matrix} C\overline{A} & B \\ & C \end{matrix}\,;\,\frac{\lambda}{\lambda-1}\right] + \delta(1-\lambda)J(B, C\overline{AB}), \\
&= \overline{AB}C(1-\lambda)\ {}_2\mathbb{P}_1\left[\begin{matrix} C\overline{A} & C\overline{B} \\ & C \end{matrix}\,;\,\lambda\right] + \delta(1-\lambda)J(B, C\overline{AB}).
\end{aligned}
$$

Proposition 1.8. *If $A, B, C \in \widehat{\mathbb{F}_q^\times}$, $A, B \neq \varepsilon$ and $A, B \neq C$, then*

(1)

$$
J(A, \overline{A}C) \cdot {}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] = J(B, \overline{B}C) \cdot {}_2\mathbb{P}_1\left[\begin{matrix} B & A \\ & C \end{matrix}\,;\,\lambda\right],
$$

$$
{}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] = {}_2\mathbb{F}_1\left[\begin{matrix} B & A \\ & C \end{matrix}\,;\,\lambda\right];
$$

(2) for $\lambda \neq 0$, 1, we have

$$
{}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] = \overline{C}(\lambda)C\overline{AB}(\lambda-1)\frac{J(B, C\overline{B})}{J(A, C\overline{A})}\,{}_2\mathbb{P}_1\left[\begin{matrix} \overline{A} & \overline{B} \\ & \overline{C} \end{matrix}\,;\,\lambda\right],
$$

$$
{}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,\lambda\right] = \overline{C}(\lambda)C\overline{AB}(\lambda-1)\frac{J(\overline{B}, \overline{C}B)}{J(A, C\overline{A})}\,{}_2\mathbb{F}_1\left[\begin{matrix} \overline{A} & \overline{B} \\ & \overline{C} \end{matrix}\,;\,\lambda\right].
$$

1.5.2. *Evaluations.*

Proposition 1.9 (Gauss summation formula)**.**

$$
{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\,;\,1\right] = \frac{B(b, c-b-a)}{B(b, c-b)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}
$$

For A, B, $C \in \widehat{\mathbb{F}_q^\times}$,

$$
{}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,1\right] = J(B, C\overline{AB}); \quad {}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}\,;\,1\right] = \frac{J(B, C\overline{AB})}{J(B, C\overline{B})}.
$$

Recall that the Gauss summation formula can be obtained from the Euler integral formula (Day 1: Equation (29)). In a like manner, its finite field version can be derived by definition.

Proposition 1.10 (Kummer)**.**
K.1 *When* $Re(b-a) > -1$,

$$ {}_2\mathbb{F}_1\left[\begin{matrix} a & c-b \\ & c \end{matrix}; -1\right] = 2^{-a}\, {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; \frac{1}{2}\right]. $$

For any characters A, B, $C \in \widehat{\mathbb{F}_q^\times}$,

$$ {}_2\mathbb{P}_1\left[\begin{matrix} A & C\overline{B} \\ & C \end{matrix}; -1\right] = \overline{A}(2)\, {}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C \end{matrix}; \frac{1}{2}\right]; \quad {}_2\mathbb{F}_1\left[\begin{matrix} A & C\overline{B} \\ & C \end{matrix}; -1\right] = \overline{A}(2)\, {}_2\mathbb{F}_1\left[\begin{matrix} A & B \\ & C \end{matrix}; \frac{1}{2}\right]. $$

K.2

$$ {}_2F_1\left[\begin{matrix} c & b \\ & c-b+1 \end{matrix}; -1\right] = {}_2F_1\left[\begin{matrix} b & c \\ & c-b+1 \end{matrix}; -1\right] = \frac{\Gamma(1+c-b)\Gamma(1+c/2)}{\Gamma(1+c)\Gamma(1-b+c/2)} $$

Let ϕ *be the quadratic character. For any characters* B, $C \in \widehat{\mathbb{F}_q^\times}$,

$$ -\, {}_2\mathbb{P}_1\left[\begin{matrix} B & C \\ & C\overline{B} \end{matrix}; -1\right] = \begin{cases} 0, & \text{if } C \text{ is not a square,} \\ \binom{B\overline{D}}{B} + \binom{B\phi\overline{D}}{B}, & \text{if } C = D^2, \end{cases} $$

for some character D*. I.e., if* $C = D^2$

$$ {}_2\mathbb{P}_1\left[\begin{matrix} B & C \\ & C\overline{B} \end{matrix}; -1\right] = B(-1)J(B\overline{D}, \overline{B}) + B(-1)J(\phi B\overline{D}, \overline{B}). $$

In terms of $\mathbb{F}$*-functions, for any characters* B, C,

$$ {}_2\mathbb{F}_1\left[\begin{matrix} B & C \\ & C\overline{B} \end{matrix}; -1\right] = 0, \text{ if } C \text{ is not a square.} $$

When $C = D^2$ *is a square, further assume* $C, B \neq \varepsilon, C\overline{B}$*, and* $C \neq B$, B^2*. Then*

$$ {}_2\mathbb{F}_1\left[\begin{matrix} B & C \\ & C\overline{B} \end{matrix}; -1\right] = \frac{\mathfrak{g}(C\overline{B})\mathfrak{g}(D)}{\mathfrak{g}(C)\mathfrak{g}(D\overline{B})} + \frac{\mathfrak{g}(C\overline{B})\mathfrak{g}(\phi D)}{\mathfrak{g}(C)g(\phi D\overline{B})}. $$

Below is a general formula which holds for all n.

Proposition 1.11 (Theorem 4.2, [17] by Greene)**.** *For any fixed* $\mathbb{F}_q$ *of characteristic* $p > 2$, A_i, $B_j \in \widehat{\mathbb{F}_q^\times}$*, and* $t \neq 0$,

$$ {}_n\mathbb{P}_{n-1}\left[\begin{matrix} A_1 & A_2 & \cdots & A_n \\ & B_2 & \cdots & B_n \end{matrix}; \frac{1}{t}\right] = A_1(-t)\left(\prod_{i=2}^n A_iB_i(-1)\right) \cdot {}_n\mathbb{P}_{n-1}\left[\begin{matrix} A_1 & A_1\overline{B_2} & \cdots & A_1\overline{B_n} \\ & A_1\overline{A_2} & \cdots & A_1\overline{A_n} \end{matrix}; t\right] $$

$$ = A_2(t)\left(\prod_{i=3}^n A_iB_i(-1)\right) \cdot {}_n\mathbb{P}_{n-1}\left[\begin{matrix} A_2\overline{B}_2 & A_2 & A_2\overline{B_3} & \cdots & A_2\overline{B_n} \\ & A_2\overline{A_1} & A_2\overline{A_3} & \cdots & A_2\overline{A_n} \end{matrix}; t\right]. $$

It corresponds to the following classical result.

Proposition 1.12. *Given* $\alpha = \{a_1, a_2, \cdots, a_n\}$, $\beta = \{1, b_2, b_3, \cdots, b_n\}$*, if we denote* $1 + a_j - \beta := \{a_j, 1 + a_j - b_2, \cdots, 1 + a_j - b_n\}$ *and* $1 + a_j - \alpha := \{1 + a_j - a_1, \cdots, 1 + a_j - a_n\}$ *then the functions* $F(\alpha, \beta; z)$ *and* $(-z)^{-a_j} F(1 + a_j - \beta, 1 + a_j - \alpha; 1/z)$ *for any* $j = 1, \ldots, n$*, satisfy the same differential equation*

$$ \left[\theta\left(\theta + b_2 - 1\right) \cdots \left(\theta + b_n - 1\right) - z\left(\theta + a_1\right) \cdots \left(\theta + a_n\right)\right] F = 0, \quad \text{where } \theta = z\frac{d}{dz}. $$

Corollary 1.13.

$$ {}_{n+1}\mathbb{P}_n \begin{bmatrix} A & B_1 & \cdots & B_n \\ & A\overline{B_1} & \cdots & A\overline{B_n} \end{bmatrix} ; \; (-1)^n \Bigg] = 0, \quad \textit{if } A \textit{ is not a square.} $$

See [27] by McCarthy for more evaluation formulas for well-posed series (see Day 1: Definition 5) at ± 1. Next we recall Lemma 1 and its proof of [22] by Li, Long and Tu in which we write the following (shifted) well-posed series at 1 as a special summation.

Lemma 1.14. *For a finite field $\mathbb{F}_q$ of odd characteristic and $A, B, C, D, E \in \widehat{\mathbb{F}_q^\times}$, we have*

$$ {}_6\mathbb{P}_5 \left[\begin{matrix} A & B & C & A & D & E \\ & A\overline{D} & A\overline{E} & \varepsilon & A\overline{B} & A\overline{C} \end{matrix} ; \; 1\right] = BCDE(-1) \sum_{t\in\mathbb{F}_q} A(t)\, {}_3\mathbb{P}_2 \left[\begin{matrix} A & B & C \\ & A\overline{D} & A\overline{E} \end{matrix} ; \; t\right]^2 $$
$$ = BCDE(-1) \sum_{t\in\mathbb{F}_q} A(t)\, {}_3\mathbb{P}_2 \left[\begin{matrix} A & D & E \\ & A\overline{B} & A\overline{C} \end{matrix} ; \; t\right]^2 . $$

Proof.

$$ BCDE(-1) \sum_{t\in\mathbb{F}_q} A(t)\, {}_3\mathbb{P}_2 \left[\begin{matrix} A & B & C \\ & A\overline{D} & A\overline{E} \end{matrix} ; \; t\right]^2 $$
$$ \overset{Prop. 1.11}{=} A(-1) \sum_{t\in\mathbb{F}_q^\times} {}_3\mathbb{P}_2 \left[\begin{matrix} A & B & C \\ & A\overline{D} & A\overline{E} \end{matrix} ; \; t\right] {}_3\mathbb{P}_2 \left[\begin{matrix} A & D & E \\ & A\overline{B} & A\overline{C} \end{matrix} ; \; 1/t\right] $$
$$ = A(-1) \sum_{t\in\mathbb{F}_q^\times} \frac{-1}{q-1} \sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi}\binom{B\chi}{A\overline{D}\chi}\binom{C\chi}{A\overline{E}\chi}\chi(t) \frac{-1}{q-1} \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} \binom{A\psi}{\psi}\binom{D\psi}{A\overline{B}\psi}\binom{E\psi}{A\overline{C}\psi}\psi(1/t) $$
$$ = A(-1) \frac{1}{(q-1)^2} \sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi}\binom{B\chi}{A\overline{D}\chi}\binom{C\chi}{A\overline{E}\chi} \sum_{\psi\in\widehat{\mathbb{F}_q^\times}} \binom{A\psi}{\psi}\binom{D\psi}{A\overline{B}\psi}\binom{E\psi}{A\overline{C}\psi} \sum_{t\in\mathbb{F}_q^\times} \chi\psi^{-1}(t) $$
$$ = A(-1) \frac{1}{q-1} \sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \binom{A\chi}{\chi}\binom{B\chi}{A\overline{D}\chi}\binom{C\chi}{A\overline{E}\chi}\binom{A\chi}{\chi}\binom{D\chi}{A\overline{B}\chi}\binom{E\chi}{A\overline{C}\chi} $$
$$ = {}_6\mathbb{P}_5 \left[\begin{matrix} A & B & C & A & D & E \\ & A\overline{D} & A\overline{E} & \varepsilon & A\overline{B} & A\overline{C} \end{matrix} ; \; 1\right]. $$

□

Exercise 1.1. Prove that in general

$$ (7) \quad {}_{2n}\mathbb{P}_{2n-1} \left[\begin{matrix} A & B_1 & \cdots & B_{n-1} & A & C_1 & \cdots C_{n-1} \\ & A\overline{C}_1 & \cdots & A\overline{C}_{n-1} & \varepsilon & A\overline{B}_1 & \cdots A\overline{B}_{n-1} \end{matrix} ; \; 1\right] $$
$$ = A^{n-1}(-1) B_1 \cdots B_{n-1} C_1 \cdots C_{n-1}(-1) \sum_{t\in\mathbb{F}_q} A(t)\, {}_n\mathbb{P}_{n-1} \left[\begin{matrix} A & B_1 & \cdots & B_{n-1} \\ & A\overline{C}_1 & \cdots & C_{n-1} \end{matrix} ; \; t\right]^2 $$

Recall that Pfaff-Saalschütz evaluation formula was proved by comparing coefficients on both sides of Euler's formula. In a parallel manner, one can obtain its finite field analogue.

Proposition 1.15 (Pfaff-Saalschütz Evaluation)**.** *For $n \in \mathbb{Z}_{>0}$,*

$$
{}_3F_2\left[\begin{matrix} a & b & -n \\ & d & 1+a+b-d-n \end{matrix}\ ;\ 1\right] = \frac{(d-a)_n(d-b)_n}{(d)_n(d-a-b)_n}.
$$

For any characters A, B, C, $D \in \widehat{\mathbb{F}_q^\times}$, we have

(8)
$$
\begin{aligned}
{}_3\mathbb{P}_2\left[\begin{matrix} A & B & C \\ & D & ABC\overline{D} \end{matrix}\ ;\ 1\right] &= J(\overline{BC}D, B)J(C, A\overline{D}) - J(D\overline{B}, AB\overline{D}) \\
&= B(-1)J(C, A\overline{D})J(B, C\overline{D}) - BD(-1)J(D\overline{B}, \overline{A}).
\end{aligned}
$$

Note also in the finite field version there is an extra term $-BD(-1)J(D\overline{B}, \overline{A})$.

Exercise 1.2.
(1) Obtain the finite field analogue of (32) of Lecture I by generalizing the idea of its proof.
(2) In this question, we will consider the finite field $\mathbb{P}$-version of Dixon's, Watson's and Whipple's ${}_3F_2$-evaluation formulas. For each of the followings, establish the $\mathbb{P}$-evaluation as well as you can.
    (a) **Dixon's evaluation:**

$$
{}_3F_2\left[\begin{matrix} a & b & c \\ & 1+a-b & 1+a-c \end{matrix}\ ;\ 1\right] = \Gamma\left(\frac{1+\frac{a}{2}, 1+a-b, 1+a-c, 1+\frac{a}{2}-b-c}{1+a, 1+\frac{a}{2}-b, 1+\frac{a}{2}-c, 1+a-b-c}\right).
$$

$$
{}_3\mathbb{P}_2\left[\begin{matrix} C & B & A \\ & A\overline{C} & A\overline{B} \end{matrix}\ ;\ 1\right] =?
$$

    (b) **Watson's evaluation:**

$$
{}_3F_2\left[\begin{matrix} a & b & c \\ & \frac{a+b+c}{2} & 2c \end{matrix}\ ;\ 1\right] = \Gamma\left(\frac{\frac{1}{2}, \frac{1}{2}+c, \frac{1+a+b}{2}, \frac{1-a-b}{2}+c}{\frac{1+a}{2}, \frac{1+b}{2}, \frac{1-a}{2}+c, \frac{1-b}{2}+c}\right).
$$

$$
{}_3\mathbb{P}_2\left[\begin{matrix} A^2C & B & C \\ & AC & B^2 \end{matrix}\ ;\ 1\right] =?
$$

    (c) **Whipple's evaluation:**

$$
{}_3F_2\left[\begin{matrix} a & 1-a & c \\ & b & 1-b+2c \end{matrix}\ ;\ 1\right] = \frac{\pi}{2^{2c-1}}\Gamma\left(\frac{b, 1-b+2c}{\frac{a+b}{2}, \frac{1+a-b}{2}+c, \frac{1-a+b}{2}, 1+c-\frac{a+b}{2}}\right).
$$

$$
{}_3\mathbb{P}_2\left[\begin{matrix} C & A & \overline{A} \\ & C^2\overline{B} & B \end{matrix}\ ;\ 1\right] =?
$$

Proposition 1.16 (Algebraic hypergeometric functions)**.**

$$
{}_2F_1\left[\begin{matrix} a & a+\frac{1}{2} \\ & \frac{1}{2} \end{matrix}\ ;\ z\right] = \frac{1}{2}\left((1+\sqrt{z})^{-2a} + (1-\sqrt{z})^{-2a}\right),
$$

[14, Theorem 8.11] *Let $A \in \widehat{\mathbb{F}_q^\times}$ have order larger than 2. Then*

$$
\begin{aligned}
{}_2\mathbb{F}_1\left[\begin{matrix} A & \phi A \\ & \phi \end{matrix}\ ;\ z\right] =&(1+\phi(z))\left(\overline{A}^2(1+\sqrt{z}) + \overline{A}^2(1-\sqrt{z})\right) \\
=&\begin{cases} 0, & \textit{if } z \textit{ is not a square}, \\ \overline{A}^2(1+\sqrt{z}) + \overline{A}^2(1-\sqrt{z}), & \textit{if } z \textit{ is a square}. \end{cases}
\end{aligned}
$$

Exercise 1.3. (1) Prove Proposition 1.16. Hint: Proposition 1.6 and the multiplication formula

$$\mathfrak{g}(\chi)\mathfrak{g}(\phi\chi) = \mathfrak{g}(\chi^2)\mathfrak{g}(\phi)\overline{\chi}(4)$$

may be helpful.

(2) Let $A \in \widehat{\mathbb{F}_q^\times}$ have order larger than 2. Show that

$$_2\mathbb{F}_1\left[\begin{matrix} A & A\phi \\ & A^2 \end{matrix}\,;\, z\right] = \left(\frac{1+\phi(1-z)}{2}\right)\left(\overline{A}^2\left(\frac{1+\sqrt{1-z}}{2}\right) + \overline{A}^2\left(\frac{1-\sqrt{1-z}}{2}\right)\right).$$

1.5.3. *Higher degree transformation formula.* A version of Clausen formula says

$$_2F_1\left[\begin{matrix} c-s-\frac{1}{2} & s \\ & c \end{matrix}\,;\,\lambda\right]^2 = {}_3F_2\left[\begin{matrix} 2c-2s-1 & 2s & c-\frac{1}{2} \\ & 2c-1 & c \end{matrix}\,;\,\lambda\right].$$

Below is a result of Evans and Greene in [13, Theorem 1.5].

Theorem 1.17 (Clausen formula)**.** *Let $C, S \in \widehat{\mathbb{F}_q^\times}$. Assume that $C \neq \phi$, and $S^2 \notin \{\varepsilon, C, C^2\}$. Then for $\lambda \neq 1$,*

$$_2\mathbb{F}_1\left[\begin{matrix} C\overline{S}\phi & S \\ & C \end{matrix}\,;\,\lambda\right]^2 = {}_3\mathbb{F}_2\left[\begin{matrix} C^2\overline{S}^2 & S^2 & C\phi \\ & C^2 & C \end{matrix}\,;\,\lambda\right] + \phi(1-\lambda)\overline{C}(\lambda)\left(\frac{J(\overline{S}^2, C^2)}{J(\overline{C},\phi)} + \delta(C)(q-1)\right).$$

Equivalently it can be stated as

Theorem 1.18 (Evans-Greene [13])**.** *Assume $\eta, K \in \widehat{\mathbb{F}_q^\times}$ such that none of $\eta, K\phi, \eta K, \eta\overline{K}$ is trivial. Suppose $\eta K = S^2$ is a square. When $t \neq 0, 1$, we have*

$$\begin{aligned}
_3\mathbb{P}_2\left[\begin{matrix} \phi & \eta & \overline{\eta} \\ & K & \overline{K} \end{matrix}\,;\,t\right] &= \phi(1-t)\left(\phi(t-1)K(t)\frac{J(\phi S, \phi K\overline{S})}{J(S, K\overline{S})}\,{}_2\mathbb{P}_1\left[\begin{matrix} \phi K\overline{S} & S \\ & K \end{matrix}\,;\,t\right]^2 - q\right) \\
&= \phi(1-t)\left({}_2\mathbb{P}_1\left[\begin{matrix} \phi K\overline{S} & S \\ & K \end{matrix}\,;\,t\right]{}_2\mathbb{P}_1\left[\begin{matrix} \phi\overline{K}S & \overline{S} \\ & \overline{K} \end{matrix}\,;\,t\right] - q\right),
\end{aligned}$$

where

$$_2\mathbb{P}_1\left[\begin{matrix} \phi K\overline{S} & S \\ & K \end{matrix}\,;\,t\right] = \overline{K}(t)\phi(t-1)\frac{J(S, K\overline{S})}{J(\phi K\overline{S}, \phi S)}\,{}_2\mathbb{P}_1\left[\begin{matrix} \phi\overline{K}S & \overline{S} \\ & \overline{K} \end{matrix}\,;\,t\right]. \tag{9}$$

When $t = 1$, we have

$$\begin{aligned}
_3\mathbb{P}_2\left[\begin{matrix} \phi & \eta & \overline{\eta} \\ & K & \overline{K} \end{matrix}\,;\,1\right] &= \phi\eta(-1)q\frac{J(\eta K, \overline{\eta}K)J(\phi, \phi K)}{J(\phi S, \overline{K})J(S, \overline{K})}\left(\frac{J(S, \phi\overline{S})}{J(S\overline{\eta}, \phi\overline{S}\eta)} + \frac{J(\phi S, \overline{S})}{J(\eta\overline{S}, \phi S\overline{\eta})}\right) \\
&= \frac{J(\eta K, \overline{\eta}K)}{J(\phi, \overline{K})}\left(J(S\overline{K}, \phi\overline{S})^2 + J(\phi S\overline{K}, \overline{S})^2\right).
\end{aligned} \tag{10}$$

In particular, when $K = \varepsilon$, this gives

$$_3\mathbb{P}_2\left[\begin{matrix} \phi & \eta & \overline{\eta} \\ & \varepsilon & \varepsilon \end{matrix}\,;\,1\right] = J(S, \phi\overline{S})^2 + J(\overline{S}, \phi S)^2. \tag{11}$$

When ηK is not a square, we have

$$_3\mathbb{P}_2\left[\begin{matrix} \phi & \eta & \overline{\eta} \\ & K & \overline{K} \end{matrix}\,;\,1\right] = 0. \tag{12}$$

Theorem 1.19 (Kummer quadratic transformation formula)**.** *Let $B, D \in \widehat{\mathbb{F}_q^\times}$, and set $C = D^2$. When $D \neq \phi$ and $B \neq D$, we have, for all $x \in \mathbb{F}_q$*

$$\overline{C}(1-x)\ {}_2\mathbb{F}_1\left[\begin{matrix} D\phi\overline{B} & D \\ & C\overline{B}\end{matrix}\ ;\ \frac{-4x}{(1-x)^2}\right]$$
$$= {}_2\mathbb{F}_1\left[\begin{matrix} B & C \\ & C\overline{B}\end{matrix}\ ;\ x\right] - \delta(1-x)\frac{J(C,\overline{B}^2)}{J(C,\overline{B})} - \delta(1+x)\frac{J(\overline{B},D\phi)}{J(C,\overline{B})}.$$

In [14], this theorem is proved using a straightforward translation of the proof over $\mathbb{C}$. It turns out the minor term of (8) plays a dedicate role in obtaining the delta terms on the right hand side, which describe the degeneracy of character sums when $x = \pm 1$.

However, do not take the similarities for granted. For example,

- Over $\mathbb{C}$.

$${}_2F_1\left[\begin{matrix} a & b \\ & c\end{matrix}\ ;\ z\right] = \Gamma\left(\begin{matrix} c, c-a-b \\ c-a, c-b\end{matrix}\right)\ {}_2F_1\left[\begin{matrix} a & b \\ & a+b+1-c\end{matrix}\ ;\ 1-z\right]$$
$$+\Gamma\left(\begin{matrix} c, a+b-c \\ a, b\end{matrix}\right)(1-z)^{c-a-b}\ {}_2F_1\left[\begin{matrix} c-a & c-b \\ & 1+c-a-b\end{matrix}\ ;\ 1-z\right],$$

- Over $\mathbb{F}_q$.

$${}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & C\end{matrix}\ ;\ \lambda\right] = B(-1)\ {}_2\mathbb{P}_1\left[\begin{matrix} A & B \\ & AB\overline{C}\end{matrix}\ ;\ 1-\lambda\right].$$

If one looks for the finite field analogue of the following useful Schwarz map in Lecture I

$$i\frac{{}_2F_1\left[\begin{matrix} \frac12 & \frac12 \\ & 1\end{matrix}\ ;\ 1-\lambda\right]}{{}_2F_1\left[\begin{matrix} \frac12 & \frac12 \\ & 1\end{matrix}\ ;\ \lambda\right]},$$

the outcome is disappointing.

Here is another example

- Over $\mathbb{C}$.

$${}_2F_1\left[\begin{matrix} a & a-\frac12 \\ & 2a\end{matrix}\ ;\ z\right] = \left(\frac{1+\sqrt{1-z}}{2}\right)^{1-2a}.$$

- Over $\mathbb{F}_q$. When $z \neq 0$,

$${}_2\mathbb{F}_1\left[\begin{matrix} A & A\phi \\ & A^2\end{matrix}\ ;\ z\right] = \begin{cases} 0 & \text{if } \phi(1-z) = -1 \\ \left(\overline{A}^2\left(\frac{1+\sqrt{1-z}}{2}\right) + \overline{A}^2\left(\frac{1-\sqrt{1-z}}{2}\right)\right) & \text{if } \phi(1-z) = 1.\end{cases}$$

Remark 1. *While ${}_nF_{n-1}$ is a single function (solution), ${}_n\mathbb{F}_{n-1}$ is by nature an average (trace) function.*

A nice example which realizes the Clausen formula geometrically is given by Ahlgren, Ono and Penniston [1]. In their work, they consider the $K3$ surfaces defined by

$$X_\lambda:\ s^2 = xy(1+x)(1+y)(x+\lambda y), \quad \lambda \neq 0, -1,$$

in relation to the elliptic curves

$$E_\lambda:\ y^2 = (x-1)(x^2 - 1/(1+\lambda)), \quad \lambda \neq 0, -1.$$

In particular, the point counting on X_λ over $\mathbb{F}_q$ is related to

$$\sum_{x,y\in\mathbb{F}_q} \phi(xy(1+x)(1+y)(x+\lambda y)) = {}_3\mathbb{P}_2\left[\begin{matrix}\phi & \phi & \phi \\ & \varepsilon & \varepsilon\end{matrix} ; -\lambda\right].$$

When $q \equiv 1 \pmod 4$, the point counting on E_λ over $\mathbb{F}_q$ (or $\mathbb{F}_{q^2}$ if needed), which is given by

$$a(\lambda, q) := -\sum_{x\in\mathbb{F}_q} \phi(x-1)\phi(x^2 - 1/(1+\lambda)),$$

is essentially ${}_2\mathbb{P}_1\left[\begin{matrix}\eta_4 & \eta_4 \\ & \varepsilon\end{matrix} ; -\lambda\right]$ where η_4 is an order 4 character. This follows from the quadratic formula in Theorem 1.19 with $B = C = \phi \in \widehat{\mathbb{F}_q^\times}$ and $x = (b+1)/(b-1)$ with $b^2 = 1 + \lambda$, for some b. Thus Theorem 1.1 of [1] is equivalent to the Clausen formula over the finite field $\mathbb{F}_q$ with $S = \eta_4$ and $C = \varepsilon$.

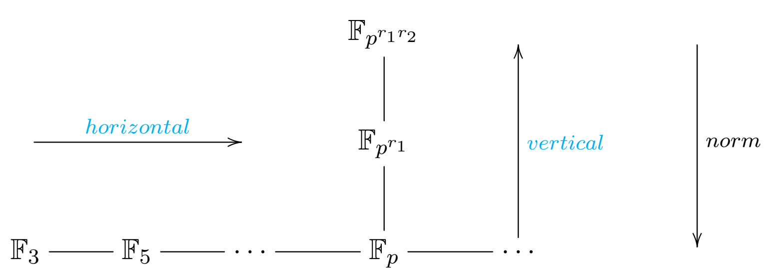
For special choices of $\lambda \in \mathbb{Q}$ such as 1, 8, 1/8, −4, −1/4, the corresponding elliptic curve E_λ has complex multiplication (CM). For these λ values, the period functions ${}_2\mathbb{P}_1$ can be written in terms of Jacobi sums and can be viewed as Grössencharacters.

1.6. **Two views of hypergeometric character sums.** For example, if $\alpha = \{\frac{1}{2}, \frac{1}{2}\}, \beta = \{1, 1\}$, for an odd prime p use

$$\frac{1}{2} \to \phi \in \widehat{\mathbb{F}_p^\times}$$

$${}_2\mathbb{P}_1\left[\begin{matrix}\phi & \phi \\ & \varepsilon\end{matrix} ; \lambda; p\right] = \sum_{x\in\mathbb{F}_p} \phi(x(1-x)(1-\lambda x))$$

in which p can be varied among all odd primes, referred to as a "horizontal" variation.



For fixed p, we can also vary the character sum "vertically" by consider the character sum induced to finite extensions $\mathbb{F}_{p^r}$ of $\mathbb{F}_p$ via the norm map. Vertical variation can be put together via

$$Z(\alpha, \beta; \lambda; p; T) = \exp\left(\sum_{r\geq 1} {}_2\mathbb{P}_1\left[\begin{matrix}\phi & \phi \\ & \varepsilon\end{matrix} ; \lambda; p^r\right] \frac{T^r}{r}\right)$$

For example, when $\lambda = -1$ and $p \equiv 1 \mod 4$, let η_4 be an order-4 character of $\widehat{\mathbb{F}_p^\times}$ and $\eta_{4,r}$ be the multiplicative character $\eta_4 \circ N_{\mathbb{F}_p}^{\mathbb{F}_{p^r}}$ for $\mathbb{F}_{p^r}^\times$. By Kummer evaluation formula (Proposition 1.10) and the vertical Hasse-Davenport relation,

$$Z(\alpha,\beta;p;T) = \exp\left(\phi(-1)\sum_{r\geq 1}(J(\eta_{4,r},\phi)+J_r(\overline{\eta_{4,r}},\phi))\frac{T^r}{r}\right)$$
$$= \exp\left(\phi(-1)\sum_{r\geq 1}(-1)^{r-1}(J(\eta_4,\phi)^r+J(\overline{\eta_4},\phi)^r)\frac{T^r}{r}\right)$$
$$= (1-\mu_pT)(1-\overline{\mu_p}T) = (1-\mu_pT)(1-p/\mu_pT)$$

where $\mu_p = -\phi(-1)J(\eta_4,\phi) = -J(\eta_4,\phi)$, since -1 is a square in $\mathbb{F}_p$.

1.7. **Another formulation when the hypergeometric data are defined over $\mathbb{Q}$.** Below we assume $\lambda \in \mathbb{Q}$ and use $lcd(\alpha,\beta;\lambda)$ to denote the least positive denominators of a_i, b_j and λ. Let $\mathbb{F}_q$ be a finite field of characteristic $p \nmid lcd(\alpha,\beta;\lambda)$. Following McCarthy, Beukers-Cohen-Mellit [3], when $lcd(\alpha,\beta;\lambda) \mid q-1$, in which $(q-1)a_j, (q-1)b_j \in \mathbb{Z}$ for all j, one defines a finite hypergeometric function over $\mathbb{F}_q$ as

$$H_q(\alpha,\beta;\lambda) := \frac{1}{1-q}\sum_{k=0}^{q-2}\prod_{j=1}^{n}\frac{\mathfrak{g}(\omega^{k+(q-1)a_j})\mathfrak{g}(\omega^{-k-(q-1)b_j})}{\mathfrak{g}(\omega^{(q-1)a_j})\mathfrak{g}(\omega^{-(q-1)b_j})}\,\omega^k\big((-1)^n\lambda\big). \tag{13}$$

When α and β are defined over $\mathbb{Q}$, write

$$\prod_{j=1}^{n}\frac{X-e^{2\pi i a_j}}{X-e^{2\pi i b_j}} = \frac{\prod_{j=1}^{r}(X^{p_j}-1)}{\prod_{k=1}^{s}(X^{q_k}-1)}$$

where $p_j, q_k \in \mathbb{Z}_{>0}$ and $p_j \neq q_k$ for all j,k. Applying the multiplication formula of Gauss sums, by Theorem 1.3 of [3], the character sum $H_q(\alpha,\beta;\lambda)$ can be rewritten as

$$H_q(\alpha,\beta;\lambda) := \frac{(-1)^{r+s}}{1-q}\sum_{m=0}^{q-2}q^{-s(0)+s(m)}\prod_{j=1}^{r}\mathfrak{g}(\omega^{mp_j})\prod_{k=1}^{s}\mathfrak{g}(\omega^{-mq_k})\omega((-1)^{q_1+\cdots+q_s}N^{-1}\lambda), \tag{14}$$

where

$$N := N(\alpha,\beta) = \frac{p_1^{p_1}\cdots p_r^{p_r}}{q_1^{q_1}\cdots q_s^{q_s}} \tag{15}$$

and $s(m)$ is the multiplicity of $X-e^{2\pi i m/(q-1)}$ in the factorization of $gcd(\prod_{j=1}^{r}(X^{p_j}-1), \prod_{k=1}^{s}(X^{q_k}-1))$.

Example 1.1. For $\alpha = \{\frac{1}{3},\frac{2}{3}\}, \beta = \{1,1\}$, $r=1, s=3, p_1=3, q_1=q_2=q_3=1$, $N=-3^3$.

Remark 2. *Note that* (13) *requires q to be congruent to 1 modulo $lcd(\alpha,\beta;\lambda)$, while this assumption is relaxed in* (14) *to as long as q is coprime to $lcd(\alpha,\beta;\lambda)$.*

Exercise 1.4. Show that under the above assumptions (13) can be written as (14).

Here we state an equivalent version of [3, Theorem 1.3]:

Proposition 1.20. *If the given HD is defined over $\mathbb{Q}$, partition the multi-set α in the form*

$$\alpha = \bigcup_{i=1}^{t}\Sigma_{d_i}, \quad \text{where } \Sigma_d = \{\ell/d : 0 < (\ell \mod d) \leq d,\ (\ell,d)=1\} \text{ for } d \in \{d_1,\ldots,d_t\}, \tag{16}$$

with $\varphi(d_1)+\cdots+\varphi(d_t)=n$ (d_i are not necessarily distinct). Let D_a be the sequence of the denominators of a_i's as

$$D_a := \{d_1,\dots,d_t\}.$$

Take D_b the sequence of the denominators of b_i's in a similar way.

Then

$$H_q(\alpha,\beta;\lambda)=\frac{1}{1-q}\sum_{k=0}^{q-2}\prod_{d\in D_a}S_d(\omega^k)\prod_{d\in D_b}S_d(\overline{\omega}^k)\omega^k\big((-1)^n\lambda\big),$$

where

$$S_d(\chi):=\prod_{m|d}\left(\frac{\mathfrak{g}(\chi^m)}{\mathfrak{g}(\chi)}\,\chi(m^{-m})\right)^{\mu(d/m)}=\frac{1}{\mathfrak{g}(\chi)^{\varphi(d)}}\prod_{m|d}\big(\mathfrak{g}(\chi^m)\,\chi(m^{-m})\big)^{\mu(d/m)},$$

and $\mu(\cdot)$ is the Möbius function.

Example 1.2. Take $HD_1=\{\{\frac13,\frac23\},\{1,1\},\lambda\}$ for example, for which case $D_a=\{3\},D_b=\{1,1\}$.

$$\begin{aligned}H_q(HD_1)=&\frac{1}{1-q}\sum_{k=0}^{q-2}\frac{\mathfrak{g}(\omega^{k+\frac{q-1}{3}})\mathfrak{g}(\omega^{k-\frac{q-1}{3}})}{\mathfrak{g}(\omega^{\frac{q-1}{3}})\mathfrak{g}(\omega^{-\frac{q-1}{3}})}\frac{\mathfrak{g}(\omega^{-k})^2}{\mathfrak{g}(\varepsilon)^2}\,\omega^k(\lambda)\\
=&\frac{1}{1-q}\sum_{k=0}^{q-2}\frac{\mathfrak{g}(\omega^{3k})}{\mathfrak{g}(\omega^k)}\mathfrak{g}(\omega^{-k})^2\,\omega^k(3^{-3}\lambda)\\
=&\frac{1}{1-q}\left(1+\frac{1}{q}\sum_{k=1}^{q-2}\mathfrak{g}(\omega^{3k})\mathfrak{g}(\omega^{-k})^3\,\omega^k(-3^{-3}\lambda)\right).\end{aligned}\tag{17}$$

For $HD_2=\{\{\frac12,\frac13,\frac23\},\{1,\frac16,\frac56\},\lambda\}$, $D_a=\{2,3\},D_b=\{1,6\}$.

$$\begin{aligned}H_q(HD_2)=&\frac{1}{1-q}\sum_{k=0}^{q-2}\frac{\mathfrak{g}(\omega^{k+\frac{q-1}{2}}\mathfrak{g}(\omega^{k+\frac{q-1}{3}})\mathfrak{g}(\omega^{k-\frac{q-1}{3}})}{\mathfrak{g}(\omega^{\frac{q-1}{2}})\mathfrak{g}(\omega^{\frac{q-1}{3}})\mathfrak{g}(\omega^{-\frac{q-1}{3}})}\frac{\mathfrak{g}(\omega^{-k-\frac{q-1}{6}})\mathfrak{g}(\omega^{-k+\frac{q-1}{6}})}{\mathfrak{g}(\omega^{\frac{q-1}{6}})\mathfrak{g}(\omega^{-\frac{q-1}{6}})}\,\omega^k(-\lambda)\\
=&\frac{1}{1-q}\sum_{k=0}^{q-2}\frac{\mathfrak{g}(\omega^{2k})\mathfrak{g}(\omega^{3k})}{\mathfrak{g}(\omega^k)^2}\frac{\mathfrak{g}(\omega^{-k})\mathfrak{g}(\omega^{-6k})}{\mathfrak{g}(\omega^{-2k})\mathfrak{g}(\omega^{-3k})}\,\omega^k(2^2\lambda).\end{aligned}$$

For convenience, when $q\equiv 1 \mod lcd(\alpha,\beta;\lambda)$ we write

$$\mathbb{P}(\alpha,\beta;\lambda;\mathbb{F}_q;\omega):={}_n\mathbb{P}_{n-1}\left[\begin{matrix}\omega^{(q-1)a_1} & \omega^{(q-1)a_2} & \dots & \omega^{(q-1)a_n}\\ & \omega^{(q-1)b_2} & \dots & \omega^{(q-1)b_n}\end{matrix}\;;\;\lambda;q\right].\tag{18}$$

Proposition 1.21. *When $\alpha=\{a_1,...,a_n\}$, $\beta=\{1,b_2,...,b_n\}$ forming a primitive pair, the H_q- and $\mathbb{P}$-functions are related by*

$$\mathbb{P}(\alpha,\beta;\lambda;\mathbb{F}_q;\omega)=H_q(\alpha,\beta;\lambda;\omega)\cdot\prod_{i=2}^{n}\omega^{(q-1)a_i}(-1)J(\omega^{(q-1)a_i},\omega^{(q-1)(-b_i)}).\tag{19}$$

and the H_q- and $\mathbb{F}$-functions are the same.

Exercise 1.5. Verify the above Proposition.

1.8. **Finite hypergeometric functions and point counts over finite fields.** Classically it is known that Jacobi sums are useful for point counts of Fermat type diagonal hypersurfaces over finite fields.

Example 1.3. Consider the elliptic curve $E: y^2 = x^3 - x$ (which is birational to the curve $y^2 = x^4 + 1/4$).

(1) When $q \equiv 3 \mod 4$,
$$\#E_n(\mathbb{F}_q) = 1 + q$$

(2) When $q \equiv 1 \mod 4$, let ϕ_q be the quadratic character and $\eta_{4,q}$ a primitive character of order 4 of $\mathbb{F}_q^\times$. Then
$$\#E(\mathbb{F}_q) = 1 + q - J(\phi_q, \eta_{4,q}) - \overline{J(\phi_q, \eta_{4,q})}.$$

In the case $p \equiv 1 \mod 4$ and $q = p^r$, by the Hasse-Davenport relation (2),
$$\#E(\mathbb{F}_q) = 1 + q - \alpha^r - \overline{\alpha}^r, \quad \alpha = J(\phi_p, \eta_{4,p}).$$

Thus, the local zeta function of E is
$$(1-T)(1-pT)Z(E/\mathbb{F}_p;T) = \begin{cases} \prod_{(\mathfrak{p})|p} (1 - \alpha_{\mathfrak{p}} T), & \text{if } p \equiv 1 \mod 4, \\ 1 + pT^2, & \text{if } p \equiv 3 \mod 4, \end{cases}$$
where $\alpha_{\mathfrak{p}} = a + bi \in \mathbb{Z}[i]$ such $\mathfrak{p} = (\alpha_{\mathfrak{p}})$ and $\alpha_{\mathfrak{p}} \equiv 1 \mod 2 + 2i$.

The elliptic curve is a CM-elliptic curve of level 32 and the associated Hecke character is given by the multiplicative character $\psi(x)$ of order 4 on $(\mathbb{Z}[i]/(2+2i))^\times$ such that $\psi(x)x \equiv 1 \mod 2 + 2i$.

The L-function of E can be written as
$$L(E,s) = \frac{1}{4} \sum_{a+bi \in \mathbb{Z}[i]} \frac{(a+bi)\psi(a+bi)}{(a^2+b^2)^s} = L(\eta(4\tau)^2\eta(8\tau)^2, s).$$

In particular,
$$L(E,1) = L(\eta(4\tau)^2\eta(8\tau)^2, 1) = \frac{1}{8} B\left(\frac{1}{2}, \frac{1}{4}\right).$$

See [21, Lemma 3.1].

Hypergeometric functions over finite fields are convenient for point counts on special algebraic varieties. We will discuss three types here.

1.8.1. Firstly, finite period functions can be used to count algebraic varieties similar to the Legendre curves $L_\lambda: \quad y^2 = x(1-x)(1-\lambda x)$. If $\lambda \in \mathbb{F}_p$.
$$\#(L_\lambda/\mathbb{F}_p) = \sum_{x \in \mathbb{F}_p} (1 + \phi(x(1-x)(1-\lambda x))) = p + {}_2\mathbb{P}_1 \left[\begin{matrix} \phi & \phi \\ & \varepsilon \end{matrix} ; \lambda\right].$$

There will be an additional 1 in the above point count formula when infinity is also considered. In general if one considers the family of hypergeometric algebraic varieties
$$X_\lambda: \quad y^N = x_1^{i_1} \cdots x_n^{i_n} \cdot (1-x_1)^{j_1} \cdots (1-x_n)^{j_n} \cdot (1 - \lambda x_1 \cdots x_n)^k,$$
then as in [8]

Proposition 1.22. *Let $q = p^e \equiv 1 \pmod N$ be a prime power, and $\eta_N \in \widehat{\mathbb{F}_q^\times}$ a primitive order N character. Then*
$$\#X_\lambda(\mathbb{F}_q) = 1 + q^n + \sum_{m=1}^{N-1} {}_{n+1}\mathbb{P}_n \left[\begin{matrix} \eta_N^{-mk} & \eta_N^{mi_n} & \cdots & \eta_N^{mi_1} \\ & \eta_N^{mi_n + mj_n} & \cdots & \eta_N^{mi_1 + mj_1} \end{matrix} ; \lambda ; q\right].$$

1.8.2. Secondly, when α, β are both defined over $\mathbb{Q}$, toric models are used in [3, Thm. 1.5] by Beukers, Cohen, Mellit (BCM). Let p_i, q_j be distinct, unique up to a re-ordering the p_i (resp. q_j), positive integers such that

$$\prod_{j=1}^{n} \frac{X - e^{2\pi i a_j}}{X - e^{2\pi i b_j}} = \frac{\prod_{j=1}^{r}(X^{p_j} - 1)}{\prod_{j=1}^{s}(X^{q_j} - 1)} \tag{20}$$

From the degrees of the rational function on the left hand side, we know

$$p_1 + \cdots + p_r = q_1 + \cdots + q_s. \tag{21}$$

The BCM varieties are defined as a subset of the torus $T = (\mathbb{C}^*)^r \times (\mathbb{C}^*)^s$ by two equations

$$V_{\alpha,\beta}(\lambda): \quad x_1 + \cdots + x_r - y_1 - \cdots - y_s = 0, \quad N\lambda x_1^{p_1} \cdots x_r^{p_r} = y_1^{q_1} \cdots y_s^{q_s}, \tag{22}$$

where $N = \frac{\prod_{i=1}^{s} q_i^{q_i}}{\prod_{i=1}^{r} p_i^{p_i}}$ as (15).

Example 1.4. For $\alpha = \{\frac{1}{3}, \frac{2}{3}\}, \beta = \{1, 1\}$, $r = 1, s = 3, p_1 = 3, q_1 = q_2 = q_3 = 1$. So the corresponding model is

$$X_1 - Y_1 - Y_2 - Y_3 = 0, \quad 27\lambda X_1^3 = Y_1 Y_2 Y_3. \tag{23}$$

Lemma 1.23 (Lemma 2.7 [3]). *Let* $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{Z}^n$. *Define* $a_{n+1} = -a_1 - \cdots - a_n$ *and* $a = \gcd(a_1, \cdots, a_n)$. *Then for any* $m \in \mathbb{Z}$

$$\sum_{v \in \mathbb{F}_q, \mathbf{x} \in (\mathbb{F}_q^\times)^n} \Phi_q(v(1 + x_1 + \cdots + x_n))\omega(\mathbf{x}^{\mathbf{a}})^m = (q-1)^n \delta(am) + \mathfrak{g}(a_1 m) \cdots \mathfrak{g}(a_{n+1} m) \tag{24}$$

The main result of [3] is as follows.

Theorem 1.24 (Beukers, Cohen, Mellit). *Let the notation* $p_i, q_j, N := N(\alpha, \beta)$ *as above. Suppose the greatest common denominators of* $p_1, \cdots, p_r, q_1, \cdots, q_s$ *is one and suppose* $N(\alpha, \beta)\lambda \neq 1$. *Then exists a suitable non-singular completion of* $V_{\alpha,\beta}(\lambda)$, *denoted by* $\overline{V_{\alpha,\beta}(\lambda)}$ *such that*

$$\#\overline{V_{\alpha,\beta}(\lambda)}/\mathbb{F}_q = P_{rs}(q) + (-1)^{r+s-1} q^{\min(r-1,s-1)} H_q(\alpha, \beta; N\lambda),$$

where

$$p_{rs}(q) = \sum_{m=0}^{\min(r-1,s-1)} \binom{r-1}{m} \binom{s-1}{m} \frac{q^{r+s-m-2} - q^m}{q-1}.$$

1.8.3. Thirdly, special Fermat type hypersurfaces with one-parameter deformations can also be counted using hypergeometric functions over finite fields. In particular, we are referring to algebraic varieties of the type $x_0^{n_0} + x_1^{n_1} + \cdots x_N^{n_N} - c\psi x_0 \cdots x_N = 0$ where c is a given constant, ψ is a parameter and $[x_0, \cdots, x_N]$ is a point in a weighted projective space, see [10] by Dolachev. One of the well-known family is called degree-N Dwork family which takes the form of

$$\mathcal{D}w_N(\psi): \quad x_0^N + x_1^N + \cdots x_{N-1}^{N-1} - N\psi x_0 \cdots x_{N-1} = 0. \tag{25}$$

For a fixed ψ, it is a $(N-2)$-dimensional projective variety with a unique up to scalar holomorphic differential $(N-2)$-form. It admits the action

$$G = \{(a_0, \cdots, a_{N-1}) \in (\mathbb{Z}/\mathrm{N}\mathbb{Z})^{N-1}, \sum a_i = 0 \mod N\}$$

via $(x_0, \cdots, x_{N-1}) \mapsto (\zeta_N^{a_0} x_0, \cdots, \zeta_N^{a_{N-1}} x_{N-1})$. A Picard-Fuchs equation of $Dw_N(\psi)$ is

$$\mathcal{L}_{\{\frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}\}, \{1, \cdots, 1\}; \psi^{-N}},$$

see an explicit computation in §6 [12] for the $N = 4$ case and [5] by Candelas, de la Ossa, Green and Parkes for the $N = 5$ case. Dwork also study the unit root functions for the Dwork family, see also [35] by Yu. Point count formulas for Dwork family over finite fields are obtained in [20] by Koblitz and [28] by McCarthy. See also [29] by Salerno for three algorithms for Dwork family.

Example 1.5. When $N = 3$, it is also known as the Hesse pencil

$$H(\psi): \quad x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2 = 0. \tag{26}$$

When $\lambda = \psi^{-3}$, a morphism from (26) to (23) is:

$$3\psi x_0 x_1 x_2 \mapsto X_1, \quad x_0^3 \mapsto Y_1, \quad x_1^3 \mapsto Y_2, \quad x_2^3 \mapsto Y_3, \tag{27}$$

which is defined over $\mathbb{Z}$.

We will next show how its point counts are related to truncated hypergeometric functions, which can be considered as Hasse invariants of the Hesse pencil.

Lemma 1.25. *Let $p > 3$ be a prime and $\psi \in \mathbb{F}_p$. Let $N_p(\psi)$ be the number of solutions of $f(\mathbf{x}; \psi) = x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2$ over $\mathbb{F}_p$, where $\mathbf{x} = (x_0, x_1, x_2)$, then*

$$\mathcal{N}_p(\psi) \equiv {}_2F_1\left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ & 1 \end{matrix}; \lambda\right]_{p-1} \mod p, \quad \textit{where } \lambda = \psi^{-3}.$$

Proof. By Fermat little theorem,

$$f(\mathbf{x}; \psi)^{p-1} \mod p = 1 - \delta_0(f(\mathbf{x})),$$

so

$$\mathcal{N}_p(\psi) = \sum_{\mathbf{x} \in \mathbb{F}_p^3} \left(1 - f(\mathbf{x}; \psi)^{p-1}\right) = p^3 - \left(\sum_{\mathbf{x}} f(\mathbf{x}; \psi)^{p-1}\right) \mod p.$$

Here $f(\mathbf{x}; \psi)^{p-1}$ is a degree $3p - 3$ homogeneous polynomial. When summing over all $\mathbf{x} \in \mathbb{F}_p^3$,

$$\sum_{\mathbf{x}} (x_1 x_2 x_3)^{p-1} = (p-1)^3 \equiv -1 \mod p,$$

while for all other monomials appear in the expansion of $f(\mathbf{x}; \psi)^{p-1}$, $\sum_{\mathbf{x}} x_1^{n_1} x_2^{n_2} x_3^{n_3} = 0 \mod p$ including $\sum_{\mathbf{x}} x_i^{3p-3} = p^2(p-1) \equiv 0 \mod p$, $i = 1, 2, 3$. Thus $\mathcal{N} \equiv C(p-1, p-1, p-1) \mod p$, where $C(p-1, p-1, p-1)$ is the coefficient of $(x_1 x_2 x_3)^{p-1}$, which is

$$\sum_i \binom{p-1}{i, i, i, p-1-3i} (-3\psi)^{p-1-3i} \equiv \sum_{i=0}^{(p-1)/3} \binom{3i}{i, i, i} (3\psi)^{-3i} \equiv {}_2F_1\left[\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ & 1 \end{matrix}; \lambda\right]_{p-1} \mod p.$$

□

The above computation is closely related a Commutative Formal Group Laws result of Stienstra in [31].

Exercise 1.6. Let $p > 3$ be a prime. Show that the number of solutions the intersection of the following two equation over $\mathbb{F}_p$ is congruent to ${}_4F_3\left[\begin{matrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ & 1 & 1 & 1 \end{matrix}; \lambda\right]$ modulo p, where $\lambda = \psi^{-6}$.

$$\begin{aligned} f_1(\psi): &\quad x_1^3 + x_2^3 + x_3^3 - 3\psi x_4 x_5 x_6 = 0, \\ f_2(\psi): &\quad x_4^3 + x_5^3 + x_6^3 - 3\psi x_1 x_2 x_3 = 0. \end{aligned} \tag{28}$$

Next we will give a more precise formula for the point count $\mathcal{N}_q(\psi)$ of Hesse pencil over any finite field $\mathbb{F}_q$ of charactersistic larger than 3 by computing its major term

$$N_q(\psi) := \frac{1}{q(q-1)} \sum_{v \in \mathbb{F}_q, \mathbf{x} \in (\mathbb{F}_q^\times)^3} \Phi_q(v(x_1^3 + x_2^3 + x_3^3 - 3\psi x_1 x_2 x_3)) \tag{29}$$

is a function from $\mathbb{F}_q \mapsto \mathbb{C}$, in which we omit the count for the cases when at least one x_i is 0. Where Φ_q denotes the additive character. Note also here we use the orthogonality of the additive character instead of the delta function so that Gauss sums will appear naturally when we apply finite Fourier analysis (or the Lagrange inversion formula). The while purpose is to show the major term can be written as the finite character sum $H_q(\{\frac{1}{3}, \frac{2}{3}\}, \{1, 1\}; \lambda)$ in the formulation of (14). We decompose the computution on the following steps.

I) Computing the inner product of $\langle N_q(\psi), \chi\rangle$ of $N_q(\psi)$ with χ when $\chi = \varepsilon$ is the trivial character.

$$\langle N_q(\psi), \varepsilon\rangle = (q-1).$$

We leave the verification as an exericse.

II) Computing $\langle N_q(\psi), \chi\rangle$ when $\chi \neq \varepsilon$.

$$(30)\quad (q-1)\langle N_q, \chi\rangle = \sum_{\psi} N_q(\psi)\bar{\chi}(\psi)$$

$$= \frac{1}{q(q-1)} \sum_{\psi} \sum_{v\in\mathbb{F}_q^\times, \mathbf{x}\in(\mathbb{F}_q^\times)^3} \Phi_q(vx_1^3)\Phi_q(vx_2^3)\Phi_q(vx_3^3)\Phi_q(-3v\psi x_1x_2x_3)\chi(\overline{-3v\psi x_1x_2x_3})\chi(-3vx_1x_2x_3)$$

$$= \frac{1}{q(q-1)}\mathfrak{g}(\overline{\chi}) \sum_{v\in\mathbb{F}_q^\times, \mathbf{x}\in(\mathbb{F}_q^\times)^3} \Phi_q(vx_1^3)\Phi_q(vx_2^3)\Phi_q(vx_3^3)\chi(-3vx_1x_2x_3)$$

Write

$$S(\chi) := \sum_{v\in\mathbb{F}_q^\times, \mathbf{x}\in(\mathbb{F}_q^\times)^3} \Phi_q(vx_1^3)\Phi_q(vx_2^3)\Phi_q(vx_3^3)\chi(-3vx_1x_2x_3).$$

Let g_3 be a primitive 3rd root of unity in $\mathbb{F}_q^\times$ (in which case $q \equiv 1 \mod 3$). Replacing x_1 by g_3x_1 is a bijection on $\mathbb{F}_q^\times$. When χ is not a cubic, then $\chi(g_3) \neq 1$. It follows $S(\chi) = 0$.

Now we assume $\chi = \eta^3$ is a cube. Then

$$\sum_{y\in\mathbb{F}_q^\times} \chi(y)\Phi_q(vy^3) = \sum_{y\in\mathbb{F}_q^\times} \eta(y^3)\Phi_q(vy^3)$$
$$= \sum_{u\in\mathbb{F}_q^\times} \eta(u)\Phi_q(vu)\left(1 + \chi_3(u) + \chi_3^2(u)\right) = \overline{\eta}(v)\mathfrak{g}(\eta) + \overline{\eta\chi_3}(v)\mathfrak{g}(\eta\chi_3) + \overline{\eta}\chi_3(v)\mathfrak{g}(\eta\overline{\chi}_3)$$

Hence,

$$\begin{aligned}
S(\chi) &= \sum_{v\in\mathbb{F}_q^\times} \chi(-3v)\left[\overline{\eta}(v)\mathfrak{g}(\eta) + \overline{\eta\chi_3}(v)\mathfrak{g}(\eta\chi_3) + \overline{\eta}\chi_3(v)\mathfrak{g}(\eta\overline{\chi}_3)\right]^3 \\
&= \sum_{v\in\mathbb{F}_q^\times} \chi(-3)\left[\mathfrak{g}(\eta) + \overline{\chi}_3(v)\mathfrak{g}(\eta\chi_3) + \chi_3(v)\mathfrak{g}(\eta\overline{\chi}_3)\right]^3 \\
&= \chi(-3)\left[\mathfrak{g}(\eta)^3 + \mathfrak{g}(\eta\chi_3)^3 + \mathfrak{g}(\eta\overline{\chi}_3)^3 + 6\mathfrak{g}(\eta)\mathfrak{g}(\eta\chi_3)\mathfrak{g}(\eta\chi_3^2)\right] \sum_{v\in\mathbb{F}_q^\times} 1 \\
&= \chi(-3)(q-1)\left[\mathfrak{g}(\eta)^3 + \mathfrak{g}(\eta\chi_3)^3 + \mathfrak{g}(\eta\overline{\chi}_3)^3 + 6\mathfrak{g}(\eta)\mathfrak{g}(\eta\chi_3)\mathfrak{g}(\eta\chi_3^2)\right] \\
&= 6q(q-1)\chi(-1)\mathfrak{g}(\chi) + \chi(-3)(q-1)\left[\mathfrak{g}(\eta)^3 + \mathfrak{g}(\eta\chi_3)^3 + \mathfrak{g}(\eta\overline{\chi}_3)^3\right]
\end{aligned}$$

In other words, when χ is a nontrivial character and $\chi=\eta^3$,
(31)

$$\langle N_q,\chi\rangle = 6\frac{q}{q-1}+\frac{\chi(-3)}{q(q-1)}\mathfrak{g}(\overline{\chi})\left[\mathfrak{g}(\eta)^3+\mathfrak{g}(\eta\chi_3)^3+\mathfrak{g}(\eta\overline{\chi}_3)^3\right] = 6\frac{q}{q-1}+\sum_{\eta,\eta^3=\chi}\frac{\eta(-3^3)}{q(q-1)}\mathfrak{g}(\overline{\eta}^3)\mathfrak{g}(\eta)^3$$

III) Applying the finite Fourier analysis (or the Lagrange inversion formula), we obtain that for $\psi\neq 0$

$$N_q(\psi)=\sum_\chi \langle N_q,\chi\rangle\chi(\psi)=\frac{1}{3}\sum_\chi\langle N_q,\chi^3\rangle\chi^3(\psi).$$

IV) The major term of the above is $\frac{1}{q(q-1)}\sum_\chi \chi^3(-3\psi)\mathfrak{g}(\overline{\chi}^3)\mathfrak{g}(\chi)^3$. From (17), we see that it can be written as $-H_q\left(\left\{\frac13,\frac23\right\},\{1,1\};\frac{1}{\lambda}\right)-\frac{1}{q}$, where $\lambda=\psi^3$.

Exercise 1.7. Try this method on the complete intersection given by the equations listed in (28).

1.8.4. *Arithmetic mirror symmetries.* There is an extensive literature regarding mirror symmetries arising from string theory in Physics, see the textbook [6] by Cox and Sheldon. We will only mention the relevant information to hypergeometric functions.

Dwork quintic threefold

$$V(\psi):=V_{\{\frac15,\frac25,\frac35,\frac45\}}(\psi):\quad X_1^5+X_2^5+X_3^5+X_4^5+X_5^5-5\psi\, X_1X_2X_3X_4X_5=0$$

has been studied extensively, for example see [4] by Candelas, de la Ossa, Rodriguez-Villegas for the arithmetic of the quintic family over finite fields. In [5, §3] by Candelas, de la Ossa, Green and Parkes, the Picard–Fuchs differential operator of $V(\psi)$ is given using variable $\lambda=\psi^{-5}$

$$\theta^4-5^{-4}\lambda(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4),\quad \text{where } \theta:=\lambda\frac{d}{d\lambda},$$

whose unique (up to scalar) holomorphic solution near zero is given by the hypergeometric function

$$\sum_{k=0}^\infty\frac{(5k)!}{k!^5}(5^{-5}\lambda)^k = {}_4F_3\left[\begin{matrix}\frac15 & \frac25 & \frac35 & \frac45\\ & 1 & 1 & 1\end{matrix};\ \lambda\right].$$

It is a hypergeometric differential equation with parameters $\alpha=\{\frac15,\frac25,\frac35,\frac45\},\beta=\{1,1,1,1\}$. There are 14 such Calabi-Yau differential equations (see a preprint by Almkvist, van Enckevort, van Straten, Zudilin). Their parameter sets are of the form $\alpha=\{r_1,1-r_1,r_2,1-r_2\},\beta=\{1,1,1,1\}$, where $r_1,r_2\in(0,1)$ and α is defined over $\mathbb{Q}$. We list the defining equations (from [26]) of the Calabi-Yau three-folds families whose Picard Fuchs operators are $\mathcal{L}_{\alpha,\beta;\lambda}$ in the following tables. The first table consists of 4 cases given by one equation. The remaining cases are given by complete intersection in weighted projective spaces. To compute their Picard-Fuchs equation, there is a general method called the Gel'fand, Zelevinskiĭ, and Kapranov (GKZ) method [15]. See [36] by Zhou for using GKZ to derive the Picard-Fuchs equation of the Hesse pencil (26), which is $\mathcal{L}_{\{\frac13,\frac23\},\{1,1\};\lambda}$.

The notion of mirror symmetry connects two types of models in String theory. See [6] on Mirror symmetry and algebraic geometry by Cox and Sheldon for more mathematical background. One of the most well-studied examples in mirror symmetry is the Dwork quintic case. For a fixed ψ, the equation $V(\psi)$ admits the action of the discrete group

$$G=\{(\zeta_5^{a_1},\dots,\zeta_5^{a_5}):a_1+\cdots+a_5\equiv 0\mod 5\}\cong(\mathbb{Z}/5\mathbb{Z})^4$$

via the map $(X_1,\dots,X_5)\mapsto(\zeta_5^{a_1}X_1,\dots,\zeta_5^{a_5}X_5)$, where $\zeta_5=e^{2\pi i/5}$ is the primitive 5-th root of unity. Its mirror threefold is constructed from the orbifold $V(\psi)/G$. One way to realize the quotient is letting $y_j=X_j^5$ for $j=1,\dots,5$, $x_1=5\psi X_1\cdots X_5$ and $\lambda=\psi^{-5}$; the image is

$$y_1+\cdots+y_5-x_1=0,\quad 5^{-5}\lambda x_1^5=y_1\cdots y_5.$$

Table 1. One-parameter families of hypersurfaces for $V_{\{r_1,r_2,1-r_1,1-r_2\}}(\psi)$

$(d_1,\dots,d_t)$	n	(r_1,r_2)	$X(n)\in\mathbb{P}^4(w_0,\dots,w_4)$	Calabi–Yau threefold equation
(5)	5	$(\frac{1}{5},\frac{2}{5})$	$X(5)\subset\mathbb{P}^4(1,1,1,1,1)$	$\sum_{j=1}^{5}X_j^5-5\psi\prod_{j=1}^{5}X_j=0$
(10)	10	$(\frac{1}{10},\frac{3}{10})$	$X(10)\subset\mathbb{P}^4(1,1,1,2,5)$	$\sum_{j=1}^{3}X_j^{10}+2X_4^5+5X_5^2-10\psi\prod_{j=1}^{5}X_j=0$
(8)	8	$(\frac{1}{8},\frac{3}{8})$	$X(8)\subset\mathbb{P}^4(1,1,1,1,4)$	$\sum_{j=1}^{4}X_j^8+4X_5^2-8\psi\prod_{j=1}^{5}X_j=0$
$(3,6)$	6	$(\frac{1}{6},\frac{1}{3})$	$X(6)\subset\mathbb{P}^4(1,1,1,1,2)$	$\sum_{j=1}^{4}X_j^6+2X_5^3-6\psi\prod_{j=1}^{5}X_j=0$

Table 2. Complete intersection of one-parameter families of hypersurfaces for $V_{\{r_1,r_2,1-r_1,1-r_2\}}(\psi)$

$(d_1,\dots,d_t)$	(r_1,r_2)	$X(n_1,\dots,n_r)$	Calabi–Yau threefold equations
$(2,2,2,2)$	$(\frac{1}{2},\frac{1}{2})$	$X(2,2,2,2)\subset\mathbb{P}^7$	$X_1^2+X_2^2-2\psi X_3X_4=0$ $X_3^2+X_4^2-2\psi X_5X_6=0$ $X_5^2+X_6^2-2\psi X_7X_8=0$ $X_7^2+X_8^2-2\psi X_1X_2=0$
$(3,3)$	$(\frac{1}{3},\frac{1}{3})$	$X(3,3)\subset\mathbb{P}^5$	$X_1^3+X_2^3+X_3^3-3\psi X_4X_5X_6=0$ $X_4^3+X_5^3+X_6^3-3\psi X_1X_2X_3=0$
$(2,2,3)$	$(\frac{1}{2},\frac{1}{3})$	$X(2,2,3)\subset\mathbb{P}^6$	$X_1^2+X_2^2+X_3^2-3\psi X_4X_5=0$ $X_4^3+X_5^3-2\psi X_1X_6X_7=0$ $X_6^2+X_7^2-2\psi X_2X_3=0$
$(2,2,4)$	$(\frac{1}{2},\frac{1}{4})$	$X(2,4)\subset\mathbb{P}^5$	$X_1^2+X_2^2+X_3^2+X_4^2-4\psi X_5X_6=0$ $X_5^4+X_6^4-2\psi X_1X_2X_3X_4=0$
(12)	$(\frac{1}{12},\frac{5}{12})$	$X(12,12)\subset\mathbb{P}^5(1,1,4,6,6,6)$	$X_1^{12}+X_2^{12}-2\psi X_5X_6=0$ $X_5^2+X_6^2+4X_3^3+6X_4^2-12\psi X_1X_2X_3X_4=0$
$(4,4)$	$(\frac{1}{4},\frac{1}{4})$	$X(4,4)\subset\mathbb{P}^5(1,1,2,1,1,2)$	$X_1^4+X_2^4+2X_3^2-4\psi X_4X_5X_6=0$ $X_4^4+X_5^4+2X_6^2-4\psi X_1X_2X_3=0$
$(4,6)$	$(\frac{1}{4},\frac{1}{6})$	$X(4,6)\subset\mathbb{P}^5(1,1,2,1,2,3)$	$X_1^4+X_2^4+2X_3^2+2X_5^2-6\psi X_4X_6=0$ $X_4^6+3X_6^2-4\psi X_1X_2X_3X_5=0$
$(3,4)$	$(\frac{1}{3},\frac{1}{4})$	$X(3,4)\subset\mathbb{P}^5(1,1,1,1,1,2)$	$X_1^3+X_2^3+X_3^3+X_4^3-4\psi X_5X_6=0$ $X_5^4+2X_6^2-3\psi X_1X_2X_3X_4=0$
$(6,6)$	$(\frac{1}{6},\frac{1}{6})$	$X(6,6)\subset\mathbb{P}^5(1,2,3,1,2,3)$	$X_1^6+2X_2^3+3X_3^2-6\psi X_4X_5X_6=0$ $X_4^6+2X_5^3+3X_6^2-6\psi X_1X_2X_3=0$
$(2,2,6)$	$(\frac{1}{2},\frac{1}{6})$	$X(2,6)\subset\mathbb{P}^5(1,1,1,1,1,3)$	$3X_1^2+X_2^2+X_3^2+X_4^2-6\psi X_1X_5=0$ $X_5^6+X_6^2-2\psi X_2X_3X_4X_6=0$

The reader may already notice that it is nothing but $V_{\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\},\{1,1,1,1\}}(\lambda)$ by formula 22. Resolving singularities, one gets a Calabi–Yau threefold $\overline{\hat{\mathcal{V}}(\lambda)}$ with generic hodge number $h^{2,1}$ equal to 1 (see [5, 6] for details). From mathematical point of view, one would ask what is the relation between the Quntic family $V_{\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\}}$ and its mirror which is of the form

$$x_1-y_1-y_2-\cdots-y_5=0,\quad \frac{1}{5^5}\lambda x_1^5=y_1\cdots y_5,\quad \lambda=\psi^{-5}.$$

Wan gave an answer by comparing their local zeta functions [32, 33]. See [11] by Doran, Kelly, Salerno, Sperber, Voight and Whitcher for discussions on arithmetic mirror symmetry on K3 surfaces.

When $\lambda = 1$, the corresponding Calabi–Yau threefold $\overline{\hat{\mathcal{V}}(1)}$ is defined over $\mathbb{Q}$ and it becomes rigid, that is, $h^{2,1}(\overline{\hat{\mathcal{V}}(1)}) = 0$ meaning that its third Betti number $B_3 = \dim H^3(\overline{\hat{\mathcal{V}}(1)}, \mathbb{C})$ is 2. It is shown by Schoen [30] that the ℓ-adic Galois representation (of the absolute Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) arising from étale cohomology $H^3_{\mathrm{et}}(\overline{\hat{\mathcal{V}}(1)}, \mathbb{Q})$ is modular in the sense that it is isomorphic to the Galois representation attached to a weight 4 level 25 Hecke eigenform $f = f_{\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\}}$, labeled 25.4.a.b in the database.

Theorem 1.26 (Dieulefait [9], Gouvêa-Yui [16])**.** *For any rigid Calabi-Yau X defined over $\mathbb{Q}$ and each prime ℓ, there is a weight 4 modular form f with integer coefficients such that the ℓ-adic Galois representation arising from the third étale cohomology group of X is isomorphic to the ℓ-adic Deligne representation associated to f.*

When $\psi = 1$, for each of the 14 families listed in Tables 1 and 2, the corresponding Calabi-Yau manifolds are rigid are defined over $\mathbb{Q}$. By the modularity theorem of Dieulefait and Gouvêa-Yui, they are all modular.

We will explain to identify the corresponding weight-4 modular forms in the next lecture.

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