Hypergeometric Functions, Character Sums and Applications, Part III

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University of Connecticut 2021 Summer Lecture Series

https://alozano.clas.uconn.edu/hypergeometric

July 1, 2021

Plan

Day 1. Hypergeometric functions over $\mathbb{C}$

- 1.I Hypergeometric functions and differential equations
- 1.II Hypergeometric formulas and Legendre curves

Day 2. Over finite fields

- 2.I Hypergeometric functions over finite fields
- 2.II Point counts over finite fields

Day 3. In Galois perspective

- 3.I Hypergeometric Galois representations
- 3.II Modularity results

Day 4. p-adic hypergeometric functions and supercongruences

- 4.I Dwork unit roots
- 4.II Supercongruences

Review

A multiset $\alpha = \{a_1, ..., a_n\}$ with $a_i \in \mathbb{Q}$ is called *defined* over $\mathbb{Q}$, if $\prod_{j=1}^{n}(X - e^{2\pi i a_j}) \in \mathbb{Z}[X]$. It is said to be *self-dual* if $\alpha \equiv -\alpha \mod \mathbb{Z}$.

A set of hypergeometric parameters consists of

$$\alpha = \{a_1, ..., a_n\}, \beta = \{b_1 = 1, b_2, ..., b_n\}$$

with $a_i, b_j \in \mathbb{Q}$. It is called *primitive* if $a_i - b_j \notin \mathbb{Z}$ for any i, j.

A hypergeometric datum is a triple

$$\{\alpha, \beta; \lambda\}$$

where $\lambda \in \mathbb{Z}$, or $\mathbb{Q}$, or $\overline{\mathbb{Q}}$.

We introduced $\mathbb{P}, \mathbb{F}, H_q$-functions over finite fields in Lecture II.

Two perspectives

$$\begin{tikzcd}
 & & & \mathbb{F}_{p^{r_1 r_2}} \arrow[d, dash] & \\
{} \arrow[rr, "\textit{horizontal}"] & & {} & \mathbb{F}_{p^{r_1}} \arrow[d, dash] & \\
\mathbb{F}_3 \arrow[r, dash] & \mathbb{F}_5 \arrow[r, dash] & \cdots \arrow[r, dash] & \mathbb{F}_p \arrow[r, dash] & \cdots \arrow[uu, "\textit{vertical}"']
\end{tikzcd}$$

We will consider now how to go horizontally in a compatible way.

Basic setup

- Degree $M := lcd(\alpha, \beta)$
- Field $K = \mathbb{Q}(\zeta_M)$
- Ring $\mathcal{O}_K = \mathbb{Z}[\zeta_M]$
- Group $G(M) := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- $\wp$ any nonzero prime ideal of $\mathbb{Z}[\zeta_M, 1/M]$
- $k_\wp := \mathcal{O}_K/\wp$, size $q(\wp) := |k_\wp| \equiv 1 \mod M$
- $\mathrm{Frob}_\wp$ the Frobenius conjugacy class of $G(M)$ at $\wp$

Notation

For a finite field $\mathbb{F}_q$ containing a primitive Mth root of 1 and any $\lambda \in \mathbb{F}_q$, recall that we write

$$\mathbb{P}(\alpha, \beta; \lambda; \mathbb{F}_q; \omega) := {}_n\mathbb{P}_{n-1} \left[\begin{matrix} \omega^{(q-1)a_1} & \omega^{(q-1)a_2} & \dots & \omega^{(q-1)a_n} \\ & \omega^{(q-1)b_2} & \dots & \omega^{(q-1)b_n} \end{matrix} ; \ \lambda; q \right].$$

Similarly

$$\mathbb{F}(\alpha, \beta; \lambda; \mathbb{F}_q; \omega) := {}_n\mathbb{F}_{n-1} \left[\begin{matrix} \omega^{(q-1)a_1} & \omega^{(q-1)a_2} & \dots & \omega^{(q-1)a_n} \\ & \omega^{(q-1)b_2} & \dots & \omega^{(q-1)b_n} \end{matrix} ; \ \lambda; q \right],$$

where $\widehat{\mathbb{F}_q^\times} = \langle \omega \rangle$.

The Legendre curves

Let $\lambda \in \mathbb{Q} \setminus \{0,1\}$ be fixed and ℓ be a fixed prime number.

$$L_\lambda : \quad y^2 = x(1-x)(1-\lambda x).$$

It gives rise to a continuous representation

$$\rho_{\lambda,\ell} : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_\ell).$$

For $p \nmid Cond(L_\lambda)$,

$$\mathrm{Tr}\rho_{\lambda,\ell}(\mathrm{Frob}_p) = p - \#(L_\lambda/\mathbb{F}_p)$$

$$\#(L_\lambda/\mathbb{F}_p) = \sum_{x\in\mathbb{F}_p}(1+\phi(x(1-x)(1-\lambda x)) = p+\mathbb{P}(\{\frac{1}{2},\frac{1}{2}\},\{1,1\};\lambda;\mathbb{F}_p;\omega)$$

It is independent of the choice of ω.

$$\begin{aligned}\mathrm{Tr}\rho_{\lambda,\ell}(\mathrm{Frob}_p) &= -\mathbb{P}(\{\frac{1}{2},\frac{1}{2}\},\{1,1\};\lambda;\mathbb{F}_p) \\ &= \phi(-1)\mathbb{F}(\{\frac{1}{2},\frac{1}{2}\},\{1,1\};\lambda;\mathbb{F}_p).\end{aligned}$$

Let ζ_M be a fixed primitive $M$the root of unity. For a prime ideal $\wp$ of $\mathbb{Z}[\zeta_M, 1/M]$, $\zeta_M \mod \wp$ in the residue field $\kappa_\wp$ of $\wp$ has order M, and it generates the cyclic group $(\kappa_\wp^\times)^{(N(\wp)-1)/M}$. Put

$$\mathbb{P}(\alpha, \beta; \lambda; \kappa_\wp) = \mathbb{P}(\alpha, \beta; \lambda; \kappa_\wp; \omega_\wp)$$

where $\omega_\wp$ is a generator of $\widehat{\kappa_\wp^\times}$ so that

$$\omega_\wp(\zeta_M \mod \wp) = \zeta_M^i, \quad i \in (\mathbb{Z}/M\mathbb{Z})^\times.$$

We choose $i = -1$ by default. Note that $\mathbb{P}(\alpha, \beta; \lambda; \kappa_\wp)$ is independent of the choice of $\omega_\wp$, but depends on the choice of i on ζ_M^i.

When $|\alpha| = |\beta| = 2$,

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = \frac{1}{B(b, c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a} dx, $$

Normalized periods on

$$ C_\lambda^{[N;i,j,k]}: \quad y^N = x^i(1-x)^j(1-\lambda x)^k, \quad \text{where} $$

$$ N = \mathrm{lcd}(a, b, c), \quad i = N \cdot (1-b), \quad j = N \cdot (1+b-c), \quad k = N \cdot a, $$

If α, β are primitive, $N \nmid i, j, k, i+j+k$, assume $0 < i, j, k < N$.
$C_\lambda^{[N;i,j,k]}$ admits an automorphism

$$ \zeta : (x, y) \mapsto (x, \zeta_N^{-1} y). $$

Let $X(\lambda)$ be its smooth model.

Jacobians and Galois representations

Use $J_\lambda^{[N;i,j,k]}$ to denote the Jacobian of $X_\lambda^{[N;i,j,k]}$. For each proper divisor d of N, $J_\lambda^{[N;i,j,k]}$ contains a factor which is isogenous to $J_\lambda^{[d;i,j,k]}$ over $\mathbb{Q}(\lambda, \zeta_N)$. Use J_λ^{prim} to denote the primitive part of $J_\lambda^{[N;i,j,k]}$, which is of dimensional $\varphi(N)$, by Archinard.

If $\lambda \in \mathbb{Q} \setminus \{0, 1\}$, there is a $2\varphi(N)$-dimensional ℓ-adic Galois representation $\rho_{\lambda,\ell}$ of $G_{\mathbb{Q}}$ arising from J_λ^{prim}. Using the action induced from

$$\zeta : (x, y) \mapsto (x, \zeta_N^{-1} y),$$

$\rho_{\lambda,\ell}|_{G(M)}$ decomposes as a direct sum of $\varphi(N)$ copies of 2-dimensional Galois representations.

Theorem (Fuselier, Long, Ramakrishna, Swisher, Tu)

Let $a, b, c \in \mathbb{Q}$ with least common denominator N such that a, b, $a - c$, $b - c \notin \mathbb{Z}$ and $\lambda \in \mathbb{Q} \setminus \{0, 1\}$. Set $K = \mathbb{Q}(\zeta_N)$ and denote its ring of integers $\mathcal{O}_K$. Let ℓ be any prime. Then there exists a representation

$$\sigma_{\lambda,\ell} : G_K := Gal(\overline{K}/K) \to GL_2(\overline{\mathbb{Q}}_\ell),$$

depending on a, b and c, that is unramified at all nonzero prime ideals $\wp$ of $\mathbb{Z}[\zeta_N, 1/N\ell]$ and satisfy $ord_\wp(\lambda) = 0 = ord_\wp(1 - \lambda)$. Furthermore, the trace of Frobenius at $\wp$ in the image of $\sigma_{\lambda,\ell}$ is the well-defined algebraic integer

$$-\mathbb{P}(\{a, b\}, \{1, c\}; \lambda; \kappa_\wp).$$

When $\varphi(N) = 2$

Let $\overline{\sigma}_{\lambda,\ell}$ be its complex conjugate, namely the requirement for the generator is changed to

$$\omega_{\wp}(\zeta_M \mod \wp) = \zeta_M.$$

Question

When does J_{λ}^{prim} admit quaternionic multiplication (QM)? Or when do $\sigma_{\lambda,\ell}$ and $\overline{\sigma}_{\lambda,\ell}$ differ by a finite order character.

It is in part motivated by a modularity theorem on 4-dimensional Galois representations of $G_{\mathbb{Q}}$ admitting QM by Atkin, Li, Liu and Long.

Theorem (Deines, Fuselier, Long, Swisher, Tu)

Let $N = 3, 4, 6$ and other notations and assumptions as above, in particular, $N \nmid i + j + k, i, j, k$. Then for each $\lambda \in \overline{\mathbb{Q}}$, $End_0(J_\lambda^{prim})$ contains a quaternion algebra over $\mathbb{Q}$ if and only if

$$B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) \Big/ B\left(\frac{k}{N}, \frac{2N-i-j-k}{N}\right) \in \overline{\mathbb{Q}}.$$

Idea of the proof: The traces of $\sigma_{\lambda,\ell}$ are $-{}_2\mathbb{P}_1 = -J \cdot {}_2\mathbb{F}_1$ functions. Due to Euler transformation, ${}_2\mathbb{F}_1$ and ${}_2\overline{\mathbb{F}}_1$ in the current context only differ by a finite character. So we want to know when $J/\overline{J}$ is a finite order character. Yamaoto's result says if it is the case then the above ratio is algebraic. Conversely, we use a result of Wüstholz.

Katz, *Exponential sums and Differential Equations*, 1990
Katz, *Another look at Dwork family*, 2009

Theorem (Katz)

Let ℓ be a prime. Given a primitive pair of multi-sets $\alpha = \{a_1, \cdots, a_n\}$, $\beta = \{1, b_2, \cdots, b_n\}$ with $M = lcd(\alpha \cup \beta)$, for any datum $HD = \{\alpha, \beta; \lambda\}$ with $\lambda \in \mathbb{Z}[\zeta_M, 1/M] \smallsetminus \{0\}$, the followings hold.

i). *There exists an ℓ-adic Galois representation $\rho_{HD,\ell} : G(M) \to GL(W_\lambda)$ unramified almost everywhere such that at each prime ideal $\wp$ of $\mathbb{Z}[\zeta_M, 1/(M\ell\lambda)]$ with norm $N(\wp) = |\kappa_\wp|$,*

$$Tr\rho_{HD,\ell}(Frob_\wp) = (-1)^{n-1}\omega_\wp^{(N(\wp)-1)a_1}(-1)\mathbb{P}(\alpha, \beta; 1/\lambda; \kappa_\wp),$$

where $Frob_\wp$ stands for the Frobenius conjugacy class of $G(M)$ at $\wp$.

iia). When $\lambda \neq 0, 1$, the dimension $d := dim_{\overline{\mathbb{Q}}_\ell} W_\lambda$ equals n and all roots of the characteristic polynomial of $\rho_{DH,\ell}(\mathrm{Frob}_\wp)$ are algebraic integers and have the same absolute value $N(\wp)^{(n-1)/2}$ under all archimedean embeddings.

iib). When $\lambda \neq 0, 1$ and HD is self-dual, then W_λ admits a symmetric (resp. alternating) bilinear pairing if n is odd (resp. even).

iii). When $\lambda = 1$, dimension d equals $n-1$. In this case if HD is self-dual, then $\rho_{HD,\ell}$ has a subrepresentation $\rho^{prim}_{HD,\ell}$ of dimension $2\lfloor \frac{n-1}{2} \rfloor$ whose representation space admits a symmetric (resp. alternating) bilinear pairing if n is odd (resp. even). All roots of the characteristic polynomial of $\rho^{prim}_{HD,\ell}(\mathrm{Frob}_{\wp})$ have absolute value $N(\wp)^{(n-1)/2}$, the same as (iia).

Recall

$$
{}_{n+1}\mathbb{F}_n \left[\begin{matrix} A_1 & A_2 & \cdots & A_{n+1} \\ & B_2 & \cdots & B_{n+1} \end{matrix} ;\ \lambda\right]
$$

$$
:= \frac{1}{\prod_{i=2}^{n+1} J(A_i, B_i\overline{A_i})} {}_{n+1}\mathbb{P}_n \left[\begin{matrix} A_1 & A_2 & \ldots & A_{n+1} \\ & B_2 & \ldots & B_{n+1} \end{matrix} ; \lambda\right].
$$

$$
|J(A_i, B_i\overline{A_i})| = \sqrt{q}, \quad \text{if } A_i, B_i, B_i\overline{A_i} \neq \varepsilon
$$

Theorem (Katz, Beukers-Cohen-Mellit)

Assumption as before and further $HD = \{\alpha, \beta; \lambda\}$ *is defined over* $\mathbb{Q}$. *Assume that exactly* m *elements in* β *are in* $\mathbb{Z}$. *Then, for each prime* ℓ, *there exists an* ℓ*-adic representation* $\rho^{BCM}_{HD,\ell}$ *of* $G_{\mathbb{Q}}$ *s.t.:*

i). $\rho^{BCM}_{HD,\ell}|_{G(M)} \cong \rho_{HD,\ell}$.

ii). *For any prime* $p \nmid \ell \cdot M$ *such that* $ord_p \lambda = 0$,

$$\begin{aligned} & Tr\, \rho^{BCM}_{HD,\ell}(Frob_p) \\ &= \phi(M, a_1)(Frob_p)\chi(\alpha, \beta; \mathbb{F}_p) H_p(\alpha, \beta; 1/\lambda) \cdot p^{(n-m)/2} \in \mathbb{Z}. \end{aligned}$$

where $\phi(M, a_1)$ *is a character of* $G_{\mathbb{Q}}$, *depending on* a_1.

iii). *When* $\lambda = 1$, $\rho^{BCM}_{HD,\ell}$ *is* $(n-1)$*-dimensional and it has a subrepresentation, denoted by* $\rho^{BCM,prim}_{HD,\ell}$, *of dimension* $2\lfloor \frac{n-1}{2} \rfloor$ *whose representation space admits a symmetric (resp. alternating) bilinear pairing if* n *is odd (resp. even). All roots of the characteristic polynomial of* $\rho^{BCM,prim}_{HD,\ell}(Frob_p)$ *have absolute value* $p^{(n-1)/2}$.

A p-adic detour to derive a step function

$$H_q(\alpha, \beta; \lambda) \\ := \frac{1}{1-q} \sum_{k=0}^{q-2} \prod_{j=1}^{n} \frac{\mathfrak{g}(\omega^{k+(q-1)a_j})\mathfrak{g}(\omega^{-k-(q-1)b_j})}{\mathfrak{g}(\omega^{(q-1)a_j})\mathfrak{g}(\omega^{-(q-1)b_j})} \omega^k((-1)^n \lambda).$$

The Gross-Koblitz formula says for integer $0 \leq k < p-1$

$$\mathfrak{g}(\omega^{-k}) = -\pi_p^k \Gamma_p \left(\frac{k}{p-1} \right),$$

where ω is the Teichmuller character of $\mathbb{F}_p^\times$, $\Gamma_p(\cdot)$ is the p-adic Gamma function, π_p is a fixed root of $x^{p-1} + p = 0$ in $\mathbb{C}_p$, where ζ_p is a primitive pth root of unity which is congruent to $1 + \pi_p$ modulo π_p^2.

A step function

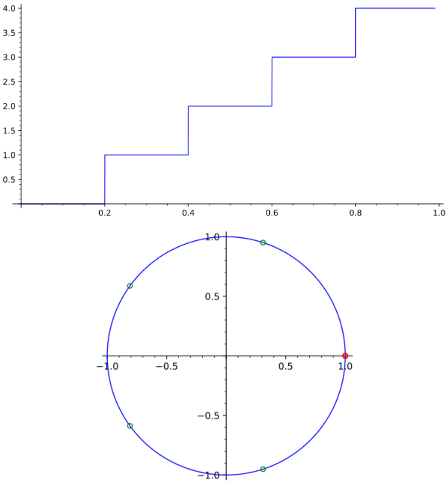
Given $\alpha = \{a_1, ..., a_n\}$ and $\beta = \{b_1, ..., b_n\}$ with $a_i, b_j \in \mathbb{Q} \cap [0, 1)$ and $M = lcd(\alpha \cup \beta)$, let

$$e_{\alpha,\beta}(x) := \sum_{i=1}^{n} - \lfloor a_i - x \rfloor - \lfloor x + b_i \rfloor . \text{ for } 0 \leq x < 1.$$

For $p \nmid lcd(\alpha, \beta)$ and $0 \leq k < p - 1$ is an integer, $e_{\alpha,\beta}(\frac{k}{p-1})$ gives the collective exponent of p in the kth summand of $H_p(\alpha, \beta; \lambda)$.

The graph of $e_{\alpha,\beta}(x)$ is a step function. The value of $e_{\alpha,\beta}(x)$ jumps up (resp. down) only at a_i (resp. $1 - b_j$). (If α, β not defined over $\mathbb{Q}$, the step function will also depends on p.)
We will compare it with the plot consisting of $\{e^{2\pi i a_j}\}_{j=1}^{n}$ and $\{e^{2\pi i b_j}\}_{j=1}^{n}$ on the unit circle mentioned in Lecture I.

$$\alpha = \{1/5, 2/5, 3/5, 4/5\}, \beta = \{0, 0, 0, 0\}$$



The weight function and the adjustment factor

- The weight $w(HD)$ of a datum $HD = \{\alpha, \beta; \lambda\}$ is defined as

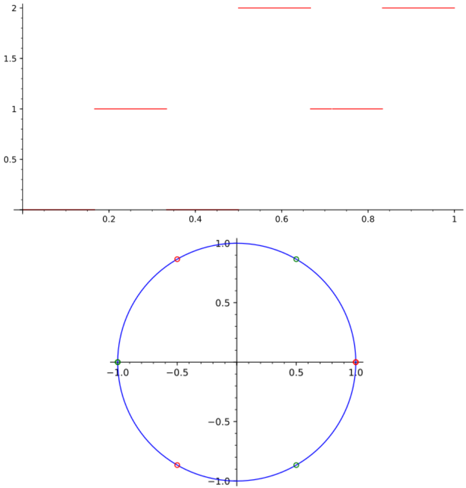
$$w(HD) := w(\alpha, \beta) := \max e_{\alpha,\beta}(x) - \min e_{\alpha,\beta}(x).$$

- The adjustment factor

$$t := -\min\{e_{\alpha,\beta}(x) \mid 0 \leq x < 1\} - \frac{n-m}{2}, \qquad (1)$$

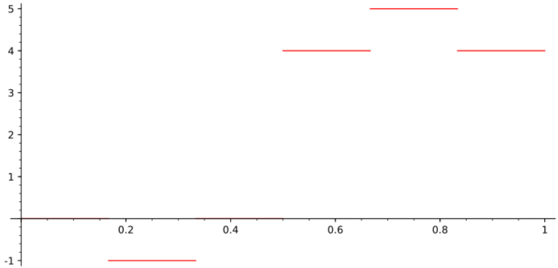
where $n = |\alpha|, m = \#\{b_j \mid b_j \in \mathbb{Z}\}$.

$\alpha = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}\}$ and $\beta = \{0, 0, \frac{1}{3}, \frac{2}{3}\}$



$max = 2, min = 0, max - min = 2, n = 4, m = 2, t = -1$

$\alpha = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$ and $\beta = \{0, 0, 0, 0, \frac{1}{6}, \frac{5}{6}\}$



$max = 5, min = -1, max - min = 6, t = -(-1) - (6-4)/2 = 0.$

Magma package implemented by Watkins

For *HD* defined over $\mathbb{Q}$, there is an efficient `Magma` program called "Hypergeometric Motives over $\mathbb{Q}$" implemented by Watkins which computes the characteristic polynomial of $\rho^{BCM}_{\{\alpha,\beta;\lambda\},\ell}[t]$ (resp. $\rho^{BCM,prim}_{\{\alpha,\beta;\lambda\},\ell}[t]$) at Fr_p, the inverse of Frob_p, for $p \nmid M\ell$ efficiently when $\lambda \neq 0, 1$ (resp. $\lambda = 1$), where $\rho[t]$ denotes the weight-t Tate twist of a representation ρ of $G_{\mathbb{Q}}$.

Example

```
H:=HypergeometricData([1/5,2/5,3/5,4/5],[1,1,1,1]);
```

$[w = 4, t = 0]$

```
Factorization(EulerFactor(H,1,7));
```

The output is $< 343 * \$.1^2 - 6 * \$.1 + 1, 1 >$

```
Factorization(EulerFactor(H,-1,7));
```

The output is

$< 117649 * \$.1^4 + 8575 * \$.1^3 + 350 * \$.1^2 + 25 * \$.1 + 1, 1 >$

$[117649 = 7^6]$

```
H2:=HypergeometricData([1/2,1/2,1/6,5/6],[0,0,1/3,2/3]);
```

$[w = 2, t = -1]$

```
Factorization(EulerFactor(H2,1,5));
```

The output is $< 5 * \$.1^2 + 2 * \$.1 + 1, 1 >$

```
Factorization(EulerFactor(H2,-1,5));
```

The output is
$< 5 * \$.1^2 - 4 * \$.1 + 1, 1 >$,
$< 5 * \$.1^2 + 2 * \$.1 + 1, 1 >$

Hypergeometric Galois representations

Note that Theorem by Katz, Beukers-Cohen-Mellit implies that one can study a whole category of Galois representations that can be explicitly computed. Roberts, Rodriguez-Villegas and Watkins use them to test standard conjectures on L-functions. For instance, according to Langlands general philosophies, these Galois representations are automorphic.

Question

When do we get degree-2 subrepresentations from hypergeometric data $HD = \{\alpha, \beta; \lambda\}$?

Here are some candidates.

- When $|\alpha| = |\beta| = 2$.
- When $|\alpha| = |\beta| = 3$, self-dual, $\lambda = 1$.
- When $|\alpha| = |\beta| = 4$, self-dual, $\lambda = 1$.
- A special construction by Li, Long and Tu using a Whipple's formula over finite field in which cases $|\alpha| = |\beta| = 6$, (well-posed), $\lambda = 1$.

Modularity theorems

Theorem

Given a prime ℓ and a 2-dimensional absolutely irreducible representation ρ of $G_{\mathbb{Q}}$ over $\overline{\mathbb{Q}}_\ell$ that is odd, unramified at almost all primes, and its restriction to a decomposition subgroup D_ℓ at ℓ is crystalline with Hodge-Tate weight $\{0, r\}$ where $1 \leq r \leq \ell - 2$ and $\ell + 1 \nmid 2r$, then ρ is modular and corresponds to a weight $r + 1$ holomorphic Hecke eigenform.

The actual identification of the target modular form can be carried out using

Theorem (Serre)

Let f be an integral weight holomorphic Hecke eigenform with coefficients in $\mathbb{Z}$. Then the p-exponents of the level of f are bounded by 8 for $p = 2$, by 5 for $p = 3$, and by 2 for all other bad primes.

Back to rigid Calabi-Yau 3folds

For $HD = \{\alpha = \{r_1, r_2, 1-r_1, 1-r_2\}, \beta = \{1,1,1,1\}; \lambda = 1\}$, where $r_1, r_2 \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ or $(r_1, r_2) = (\frac{1}{5}, \frac{2}{5})$, $(\frac{1}{10}, \frac{3}{10}), (\frac{1}{8}, \frac{3}{8}), (\frac{1}{12}, \frac{5}{12})$. For these, $\rho_{HD,\ell}^{BCM}$ of $G_{\mathbb{Q}}$ are 3-dimensional, which decomposes into a direct sum of 2 subrepresentations $\rho_{HD,\ell}^{BCM,prim} \oplus \rho_{HD,\ell}^{BCM,1}$. Among them $\rho_{HD,\ell}^{BCM,prim}$ is 2-dimensional and $\rho_{HD,\ell}^{BCM,1}$ is 1-dimensional.

Theorem (Long, Tu, Yui and Zudilin)

Let $p > 5$ be a prime and α and β as above. Then the following equality holds:

$$H_p(\boldsymbol{\alpha}, \boldsymbol{\beta}; 1) = a_p(f_{\boldsymbol{\alpha}}) + \chi_{\boldsymbol{\alpha}}(p) \cdot p,$$

where $a_p(f_{\boldsymbol{\alpha}})$ is the p-th coefficient of the normalized Hecke eigenform and $\chi_{\boldsymbol{\alpha}}$ is a Dirichlet character of order at most 2, whose precise description is given in the following table.

(r_1, r_2)	$f_\alpha(\tau)$	level	LMFDB label	$\chi_{\boldsymbol{\alpha}}$
$(\frac{1}{2}, \frac{1}{2})$	$\eta_2^4\eta_4^4$	$8 = 2^3$	`8.4.a.a`	χ_1
$(\frac{1}{2}, \frac{1}{3})$		$36 = 2^2 \cdot 3^2$	`36.4.a.a`	χ_3
$(\frac{1}{2}, \frac{1}{4})$	$\eta_4^{16}/(\eta_2^4\eta_8^4)$	$16 = 2^4$	`16.4.a.a`	χ_2
$(\frac{1}{2}, \frac{1}{6})$			`72.4.a.b`	χ_1
$(\frac{1}{3}, \frac{1}{3})$		$27 = 3^3$	`27.4.a.a`	χ_1
$(\frac{1}{3}, \frac{1}{4})$	η_3^8	$9 = 3^2$	`9.4.a.a`	χ_6
$(\frac{1}{3}, \frac{1}{6})$		$108 = 2^2 \cdot 3^3$	`108.4.a.a`	χ_3
$(\frac{1}{4}, \frac{1}{4})$			`32.4.a.a`	χ_1
$(\frac{1}{4}, \frac{1}{6})$			`144.4.a.f`	χ_2
$(\frac{1}{6}, \frac{1}{6})$		$216 = 2^3 \cdot 3^3$	`216.4.a.c`	χ_1
$(\frac{1}{5}, \frac{2}{5})$		$25 = 5^2$	`25.4.a.b`	χ_5
$(\frac{1}{8}, \frac{3}{8})$			`128.4.a.b`	χ_2
$(\frac{1}{10}, \frac{3}{10})$		$200 = 2^3 \cdot 5^2$	`200.4.a.f`	χ_1
$(\frac{1}{12}, \frac{5}{12})$		$864 = 2^5 \cdot 3^3$	`864.4.a.a`	χ_1

Other formulas in light of Galois representations

Clausen formula

$$ {}_2F_1\left[\begin{matrix} c-s-\frac{1}{2} & s \\ & c \end{matrix}\,;\ \lambda\right]^2 = {}_3F_2\left[\begin{matrix} 2c-2s-1 & 2s & c-\frac{1}{2} \\ & 2c-1 & c \end{matrix}\,;\ \lambda\right]. $$

Theorem (Evans-Greene)

Let $C, S \in \widehat{\mathbb{F}_q^\times}$. Assume that $C \neq \phi$, and $S^2 \notin \{\varepsilon, C, C^2\}$. Then for $\lambda \neq 1$,

$$ {}_2\mathbb{F}_1\left[\begin{matrix} C\overline{S}\phi & S \\ & C \end{matrix}\,;\ \lambda\right]^2 = {}_3\mathbb{F}_2\left[\begin{matrix} C^2\overline{S}^2 & S^2 & C\phi \\ & C^2 & C \end{matrix}\,;\ \lambda\right] + \phi(1-\lambda)\overline{C}(\lambda)\left(\frac{J(\overline{S}^2, C^2)}{J(\overline{C}, \phi)} + \delta(C)(q-1)\right). $$

Whipple's formula

Theorem (Whipple)

$$
{}_7F_6\left[\begin{matrix} a & 1+\frac{a}{2} & c & d & e & f & g \\ & \frac{a}{2} & 1+a-c & 1+a-d & 1+a-e & 1+a-f & 1+a-g \end{matrix}; 1\right]
$$
$$
= C \cdot {}_4F_3\left[\begin{matrix} a & e & f & g \\ & e+f+g-a & 1+a-c & 1+a-d \end{matrix}; 1\right],
$$

when both sides terminate

- The parameter set of ${}_7F_6(1)$ is imprimitive
- The ${}_7F_6(1)$ is well-posed, meaning the upper and lower parameters sum to $1+a$ in each column. Namely $\alpha = \{a_1, \cdots, a_7\}$, $\beta = \{1+a-a_i, i = 1, \cdots, 7\}$
- It is widely known and used

Whipple's formula + self-dual

$$
{}_7F_6\left[\begin{matrix} a & 1+\frac{a}{2} & c & d & e & f & g \\ & \frac{a}{2} & 1+a-c & 1+a-d & 1+a-e & 1+a-f & 1+a-g \end{matrix}; 1\right]
$$
$$
= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-e-f)\Gamma(1+a-e-g)} \times
$$
$$
{}_4F_3\left[\begin{matrix} a & e & f & g \\ & e+f+g-a & 1+a-c & 1+a-d \end{matrix}; 1\right],
$$

- Goal: to make ${}_6F_5(1)$ reduced from ${}_7F_6(1)$ self-dual
- α being self-dual, "means" if $a \in \alpha$, $1-a$ is also in α.
  $c + d = 1, \quad f + g = 1$
- $\beta = \{1 + a - a_i\}$ being self-dual requires $a = \frac{1}{2} \Rightarrow e = \frac{1}{2}$.

$$
a = \frac{1}{2}, \quad c + d = 1, \quad f + g = 1, \quad \text{and } e = \frac{1}{2}\left(-\frac{p}{2}\right)
$$

$$
{}_7F_6\left[\begin{matrix}\frac{1}{2} & \frac{5}{4} & c & 1-c & \frac{1-p}{2} & f & 1-f \\ & \frac{1}{4} & \frac{3}{2}-c & \frac{1}{2}+c & 1+\frac{p}{2} & \frac{3}{2}-f & \frac{1}{2}+f\end{matrix} ;\ 1\right] =
$$

$$
C \times \left(p \cdot {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1-p}{2} & f & 1-f \\ & 1-\frac{p}{2} & \frac{3}{2}-c & \frac{1}{2}+c\end{matrix} ;\ 1\right]\right),
$$

$C = \frac{\Gamma(\frac{p}{2})\Gamma(\frac{3}{2}-f)\Gamma(\frac{1}{2}+f)\Gamma(\frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(1+\frac{p}{2}-f)\Gamma(\frac{p}{2}+f)}$. Ignoring p, and canceling $\frac{5}{4}$ and $\frac{1}{4}$ which correspond to the same character in $\mathbb{F}_q$.

L.H.S.

$$
HD_1 = \{\alpha_6(c,f) := \left\{\frac{1}{2}, c, 1-c, \frac{1}{2}, f, 1-f\right\},
$$

$$
\beta_6(c,f) = \left\{1, \frac{3}{2}-c, \frac{1}{2}+c, 1, \frac{3}{2}-f, \frac{1}{2}+f\right\}; 1\}
$$

R.H.S.

$$
HD_2 = \left\{\alpha_4(f) := \left\{\frac{1}{2}, \frac{1}{2}, f, 1-f\right\}, \beta_4(c) := \left\{1, 1, \frac{3}{2}-c, \frac{1}{2}+c\right\}; 1\right\}.
$$

Whipple's formula in terms of Galois representations

For $(c, f) \in \mathbb{Q}^2$ s.t. HD_1, HD_2 both primitive.
Let $M(c,f) := lcd(HD_2), N(c,f) := lcd(\frac{1+2f-2c}{4}, \frac{3-2f-2c}{4})$.
$HD_1 \mapsto \rho_{HD_1(c,f),\ell}$ of $G(M)$ which is $6-1 = 5 = 4+1$ dim'l.
$HD_2 \mapsto \rho_{HD_2(c,f),\ell}$ of $G(M)$ which is $4-1 = 3 = 2+1$ dim'l.

Theorem (Li, L. Tu)

Given any prime ℓ,

$$\rho_{HD_1(c,f),\ell}|_{G(N(c,f))} \cong (\epsilon_\ell \otimes \rho_{HD_2(c,f),\ell})|_{G(N(c,f))} \oplus \sigma_{sym,\ell}$$

where ϵ_ℓ is the ℓ-adic cyclotomic character, and $\sigma_{sym,\ell}$ is a 2-dimensional representation of $G(N)$ that can be computed explicitly.

- Over $\mathbb{C}$.

$$
{}_7F_6\left[\begin{matrix}\frac{1}{2} & \frac{5}{4} & c & 1-c & \frac{1-p}{2} & f & 1-f \\ & \frac{1}{4} & \frac{3}{2}-c & \frac{1}{2}+c & 1+\frac{p}{2} & \frac{3}{2}-f & \frac{1}{2}+f\end{matrix};\ 1\right] = C \times \left(p \cdot {}_4F_3\left[\begin{matrix}\frac{1}{2} & \frac{1-p}{2} & f & 1-f \\ & 1-\frac{p}{2} & \frac{3}{2}-c & \frac{1}{2}+c\end{matrix};\ 1\right]\right),
$$

- In Galois perspective.

Theorem (Li, Long, Tu)

Given any prime ℓ,

$$
\rho_{HD_1(c,f),\ell}|_{G(N(c,f))} \cong \left(\epsilon_\ell \otimes \rho_{HD_2(c,f),\ell}\right)|_{G(N(c,f))} \oplus \sigma_{sym,\ell}
$$

where ϵ_ℓ is the ℓ-adic cyclotomic character ($\epsilon_\ell(Frob_p) = p$, when $p \neq \ell$), and $\sigma_{sym,\ell}$ is a 2-dimensional representation of $G(N)$ that can be computed explicitly.

Theorem (Li, Long, and Tu)

For each pair (c, f) in the list, $\rho^{BCM}_{HD_1(c,f),\ell}$ is modular (using LMFDB label).

(c, f)	$Tr\, \rho^{BCM}_{HD_1(c,f),\ell}(Frob_p)$
$(\frac{1}{2}, \frac{1}{2})$	$a_p(f_{8.6.a.a}) + p \cdot a_p(f_{8.4.a.a}) + \left(\frac{-1}{p}\right) p^2$
$(\frac{1}{2}, \frac{1}{3})$	$a_p(f_{4.6.a.a}) + p \cdot a_p(f_{12.4.a.a}) + \left(\frac{3}{p}\right) p^2$
$(\frac{1}{3}, \frac{1}{3})$	$a_p(f_{6.6.a.a}) + p \cdot a_p(f_{18.4.a.a}) + \left(\frac{-1}{p}\right) p^2$
$(\frac{1}{2}, \frac{1}{6})$	$p \cdot a_p(f_{8.4.a.a}) + p \cdot a_p(f_{24.4.a.a}) + \left(\frac{3}{p}\right) p^2$
$(\frac{1}{6}, \frac{1}{6})$	$p^2 \cdot a_p(f_{24.2.a.a}) + p^2 \cdot a_p(f_{72.2.a.a}) + \left(\frac{-1}{p}\right) p^2$
$(\frac{1}{5}, \frac{2}{5})$	$p \cdot a_p(f_{10.4.a.a}) + p \cdot a_p(f_{50.4.a.d}) + \left(\frac{-5}{p}\right) p^2$
$(\frac{1}{10}, \frac{3}{10})$	$p^2 \cdot a_p(f_{40.2.a.a}) + p^2 \cdot a_p(f_{200.2.a.b}) + \left(\frac{-5}{p}\right) p^2$

$(c, f) = (\frac{1}{2}, \frac{1}{2})$, supercongruences for each odd prime p

$H_p(HD_1) = a_p(f_{8.6.a.a}) + p \cdot a_p(f_{8.4.a.a}) + \left(\frac{-1}{p}\right) p^2$. First was conjectured by Koike and was shown by Frechette-Ono-Papanikolas.

Mortenson conjectured that for each odd prime p

$$ {}_6F_5\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 & 1 & 1 \end{matrix} ; \; 1\right]_{p-1} \overset{?}{\equiv} a_p(f_{8.6.a.a}) \mod p^5. $$

Mod p^3 version was proved by Osburn-Straub-Zudilin.

$(c, f) = (\frac{1}{2}, \frac{1}{2})$, Archmidean version

$$ {}_6F_5\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 & 1 & 1 \end{matrix} ;\ 1\right]_{p-1} \overset{?}{\equiv} a_p(f_{8.6.a.a}) \mod p^5. $$

$$ {}_7F_6\left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 & 1 & 1 & 1 \end{matrix} ;\ 1\right]_{p-1} \overset{?}{\equiv} p \cdot a_p(f_{8.4.a.a}) \mod p^4. $$

Theorem (Li, L. Tu)

$$ {}_6F_5\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 & 1 & 1 & 1 \end{matrix} ;\ 1\right] = 16 \int_{1/2+i/2}^{-1/2+i/2} \tau^2 f_{8.6.a.a}\left(\frac{\tau}{2}\right) d\tau $$

$$ {}_7F_6\left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 & 1 & 1 & 1 \end{matrix} ;\ 1\right] = \frac{32i}{\pi} \int_{1/2+i/2}^{-1/2+i/2} \tau f_{8.4.a.a}(\frac{\tau}{2}) d\tau, $$

where the path is the hyperbolic geodesic from $\frac{1+i}{2}$ to $\frac{-1+i}{2}$, clockwise.

Thank you!

Appendix

Theorem (Wüstholz)

Let A be an abelian variety isogenous over $\overline{\mathbb{Q}}$ to the direct product $A_1^{n_1} \times \cdots \times A_k^{n_k}$ of simple, pairwise non-isogenous abelian varieties A_μ defined over $\overline{\mathbb{Q}}$, $\mu = 1, \ldots, k$. Let $\Lambda_{\overline{\mathbb{Q}}}(A)$ denote the space of all periods of differentials, defined over $\overline{\mathbb{Q}}$, of the first kind and the second on A. Then the vector space $\widehat{V}_A$ over $\overline{\mathbb{Q}}$ generated by 1, $2\pi i$, and $\Lambda_{\overline{\mathbb{Q}}}(A)$, has dimension

$$\dim_{\overline{\mathbb{Q}}} \widehat{V}_A = 2 + 4 \sum_{\nu=1}^{k} \frac{\dim A_\nu^2}{\dim_{\mathbb{Q}}(End_0 A_\nu)},$$

where $End_0(A_\nu) = End(A_\nu) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Example

When $\dim A = 2$, $\dim_{\overline{\mathbb{Q}}} \widehat{V}_A = 4, 6, 10$.