Hypergeometric Functions, Character Sums and Applications, Part IV

Ling Long
Notes by Ling Long and Fang-Ting Tu
University of Connecticut 2021 Summer Lecture Series

https://alozano.clas.uconn.edu/hypergeometric

July 2, 2021

Plan

Day 1. Hypergeometric functions over $\mathbb{C}$

- 1.I Hypergeometric functions and differential equations
- 1.II Hypergeometric formulas and Legendre curves

Day 2. Over finite fields

- 2.I Hypergeometric functions over finite fields
- 2.II Point counts over finite fields

Day 3. In Galois perspective

- 3.I Hypergeometric Galois representations
- 3.II Modularity results

Day 4. p-adic hypergeometric functions and supercongruences

- 4.I Dwork unit roots
- 4.II Supercongruences

Morita p-adic Gamma function

Recall that the Mortita p-adic Gamma function $\Gamma_p(x)$ is defined for $n \in \mathbb{N}$ by

$$\Gamma_p(n) := (-1)^n \prod_{\substack{0<j<n \\ p \nmid j}} j,$$

and extends to $x \in \mathbb{Z}_p$ by defining $\Gamma_p(0) := 1$, and for $x \neq 0$,

$$\Gamma_p(x) := \lim_{n \to x} \Gamma_p(n),$$

where n runs through any sequence of positive integers p-adically approaching x.

Theorem

1). $\Gamma_p(0) = 1$

2). $\dfrac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & \textit{if } |x|_p = 1 \\ -1 & \textit{if } |x|_p < 1. \end{cases}$

3). $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{a_0(x)}$ *where* $a_0(x) \in \{1, 2, \cdots, p\}$ *satisfies* $x - a_0(x) \equiv 0 \mod p$.

4). $\Gamma_p\left(\dfrac{1}{2}\right)^2 = (-1)^{\frac{p+1}{2}}$.

There are also multiplication formulas for Γ_p.

Analytic properties of $\Gamma_p(\cdot)$

Theorem (Morita, Barsky)

For $a \in \mathbb{Z}_p$ the function $x \mapsto \Gamma_p(a+x)$ is locally analytic on $\mathbb{Z}_p$ and converges for $v_p(x) \geq \frac{1}{p} + \frac{1}{p-1}$.

Recall the p-adic logarithm

$$\log_p(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n},$$

which converges for $x \in \mathbb{C}_p$ with $|x|_p < 1$.
Set

$$G_k(a) = \Gamma_p^{(k)}(a)/\Gamma_p(a). \tag{1}$$

In particular, $G_0(a) = 1$.

Lemma

For a in $\mathbb{Z}_p$, $G_1(a) = G_1(1-a)$, and $G_2(a) + G_2(1-a) = 2G_1^2(a)$.

G_1 and partial harmonic sums

Proposition

Let $x \in \mathbb{Z}_p^\times$. Than

(1) $G_1(x+1) - G_1(x) = 1/x$.

(2) $G_2(x+1) - G_2(x) = G_1(x+1)^2 - G_1(x) - 1/x^2$.

Corollary

If $1, 2, \cdots, k \in \mathbb{Z}_p^\times$, then

$$G_1(k+1) - G_1(1) = \sum_{j=1}^{k} \frac{1}{j} = H_k,$$

the partial Harmonic sum. Similarly, if $\frac{1}{2}, \frac{3}{2}, \cdots, k + \frac{1}{2} \in \mathbb{Z}_p^\times$, then

$$G_1\left(k + \frac{1}{2}\right) - G_1\left(\frac{1}{2}\right) = \sum_{j=1}^{k} \frac{1}{2j-1}.$$

Theorem

For $p \geq 5$, $r \in \mathbb{N}$, $a \in \mathbb{Z}_p, m \in \mathbb{C}_p$ satisfying $v_p(m) \geq 0$ and $t \in \{0, 1, 2\}$ we have

$$\frac{\Gamma_p(a + mp^r)}{\Gamma_p(a)} \equiv \sum_{k=0}^{t} \frac{G_k(a)}{k!}(mp^r)^k \mod p^{(t+1)r}.$$

The above result also holds for $t = 4$ if $p \geq 11$.

— the map $\mathbb{Q} \cap \mathbb{Z}_p \to \mathbb{Q} \cap \mathbb{Z}_p$ defined by

$$r' = (r + [-r]_0)/p, \tag{2}$$

where $[a]_0 \in [0, 1, \cdots, p-1]$ and $[a]_0 \equiv a \mod p$, i.e. the first p-adic digit of $a \in \mathbb{Z}_p$.

By definition $1' = (1 + (p-1))/p = 1$, $\frac{1}{2}' = (\frac{1}{2} + \frac{p-1}{2})/p = \frac{1}{2}$, if p odd.

==

Despite the appearance, it has nothing to do with the usual derivative.

Converting Gamma quotients into p-adic Gamma quotients

Lemma

Let $a \in (0,1] \cap \mathbb{Q}$.

1) *If $v_p(a) = 0$ then $\forall m, r \in \mathbb{N}$,*

$$\frac{\Gamma(a+mp^r)}{\Gamma(a+mp^{r-1})} = (-1)^m p^{mp^{r-1}} \frac{\Gamma_p(a+mp^r)}{\Gamma_p(a)} \frac{(a')_{mp^{r-1}}}{(a)_{mp^{r-1}}},$$

2) *Suppose $a + mp^r \in \mathbb{N}$, $\forall r \in \mathbb{N}$. Then*

$$\frac{\Gamma(a+mp^r)}{\Gamma(a+mp^{r-1})} = (-1)^{a+mp^r} p^{a+mp^{r-1}-1} \Gamma_p(a+mp^r).$$

3) *Let $a, b \in \mathbb{Q}$ and suppose $a - b \in \mathbb{Z}$ and $a, b \notin \mathbb{Z}_{\leq 0}$. If none of the numbers between a and b that differ from both by an integer are divisible by p then $\dfrac{\Gamma(a)}{\Gamma(b)} = (-1)^{a-b} \dfrac{\Gamma_p(a)}{\Gamma_p(b)}$.*

From character sums to truncated hypergeometric series

In Lecture III, we recalled how to use a Magma package implemented by Watkins to evaluate hypergeometric character sums when the hypergeometric datum is defined over $\mathbb{Q}$.

$$H_q(\alpha, \beta; \lambda; \omega) := \frac{1}{1-q} \sum_{k=0}^{q-2} \prod_{j=1}^{n} \frac{\mathfrak{g}(\omega^{k+(q-1)a_j})\mathfrak{g}(\omega^{-k-(q-1)b_j})}{\mathfrak{g}(\omega^{(q-1)a_j})\mathfrak{g}(\omega^{-(q-1)b_j})} \omega^k((-1)^n\lambda).$$

Theorem (Gross and Koblitz)

For integer $0 \le k < p-1$

$$\mathfrak{g}(\omega^{-k}) = -\pi_p^k \Gamma_p\left(\frac{k}{p-1}\right), \tag{3}$$

where ω *is the Teichmuller character of* $\mathbb{F}_p^\times$, $\Gamma_p(\cdot)$ *is the p-adic Gamma function,* π_p *is a fixed root of* $x^{p-1} + p = 0$ *in* $\mathbb{C}_p$, *where* ζ_p *is a primitive pth root of unity which is congruent to* $1 + \pi_p$ *modulo* π_p^2.

An example

Theorem (Long, Tu, Yui and Zudilin)

Let $HD = \{\alpha = \{r_1, r_2, 1 - r_1, 1 - r_2\}, \beta = \{1, 1, 1, 1\}; \lambda = 1\}$, *where* $r_1, r_2 \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ *or* $(r_1, r_2) = (\frac{1}{5}, \frac{2}{5}), (\frac{1}{10}, \frac{3}{10}), (\frac{1}{8}, \frac{3}{8}), (\frac{1}{12}, \frac{5}{12})$*. Let* $p > 5$ *be a prime. Then :*

$$H_p(\boldsymbol{\alpha}, \boldsymbol{\beta}; 1) = a_p(f_{\boldsymbol{\alpha}}) + \chi_{\boldsymbol{\alpha}}(p) \cdot p,$$

where $f_{\boldsymbol{\alpha}}$ *has weight-4.*

Corollary

Notation as above,

$${}_4F_3\left[\begin{matrix} r_1 & r_2 & 1 - r_1 & 1 - r_2 \\ & 1 & 1 & 1 \end{matrix}\; ;\; 1\right]_{p-1} \equiv a_p(f_{\boldsymbol{\alpha}}) \mod p.$$

Conjecture (Rodriguez-Villegas)

The above congruence holds when modulo p^3*.*

Dwork's (generic) congruences

Theorem (Dwork)

Let p be a fixed prime, $\alpha = \{a_1, \cdots, a_n\}$ with $a_i \in \mathbb{Q}$, $\beta = \{1, \cdots, 1\}$. Assume $p \nmid lcd(\alpha)$ and denote $\{a_1', \cdots, a_n'\}$ by α'. Let $F_m(\alpha, \beta; x) := \sum_{k=0}^{m} A(k)x^k$ which is a polynomial in $\mathbb{Z}_p[x]$ where $A(k) = \prod_{i=1}^{n} \frac{(a_i)_k}{(b_i)_k} = \prod_{i=1}^{n} \frac{(a_i)_k}{k!}$ then for any positive integers s, t, m satisfying $t \geq s$

$$\begin{aligned} &F_{mp^t-1}(\alpha, \beta; x) F_{mp^{s-1}-1}(\alpha', \beta; x^p) \\ &\qquad \equiv F_{mp^{t-1}-1}(\alpha', \beta; x^p) F_{mp^s-1}(\alpha, \beta; x) \mod p^s \mathbb{Z}_p[[x]]. \end{aligned}$$

Dwork unit root functions

Near $x \in \mathbb{Z}_p$ such that $F_{p-1}(\alpha, \beta; x) \neq 0 \mod p$, which is called the ordinary case, the quotient

$$\gamma_{\alpha,p}(x) := \lim_{s \to \infty} F_{mp^s-1}(\alpha, \beta; x)/F_{mp^{s-1}-1}(\alpha', \beta; x^p) \qquad (4)$$

is a p-adic convergent function, referred to as the Dwork unit root functions.

Dwork inspired many results, including unit root functions for Dwork family by Yu and recent papers on Dwork Crystals by Beukers and Vlasenko.

The role of $\gamma_{\alpha,p}(\lambda)$

For the 14 one-parameter family of hypergeometric Calabi-Yau 3-folds, they can be given explicitly by algebraic equations.

$$V_{\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\}}(\psi): \quad X_1^5+X_2^5+X_3^5+X_4^5+X_5^5-5\psi X_1X_2X_3X_4X_5=0$$

$$V_{\{\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\}}(\psi): \quad \begin{aligned} X_1^3+X_2^3+X_3^3-3\psi X_4X_5X_6=0 \\ X_4^3+X_5^3+X_6^3-3\psi X_1X_2X_3=0 \end{aligned}$$

Using a formal group law theorem of Stienstra, it can be shown that in the ordinary case, $\gamma_{\alpha,p}(\lambda)$ is a reciprocal root of the numerator of the local Zeta function over $\mathbb{F}_p$ of $V_\alpha(\psi)$ computed from $H^3_{cris}(V_\alpha)(\psi)\otimes\mathbb{Q}$. Here $\lambda=\psi^{-5}$ and $\lambda=\psi^{-6}$ respectively. When $\psi=1$, together with modularity result, this means $\gamma_{\alpha,p}(1)$ is a root of $T^2-a_p(f_\alpha)T+p^3=0$ in $\mathbb{Z}_p^\times$. Consequently,

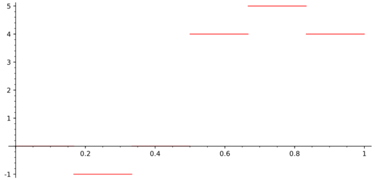
$$\gamma_{\alpha,p}(1)\equiv a_p(f_\alpha) \mod p^3$$

for all ordinary primes $p>5$.

Remark

- *If* $\alpha = \{a_1, \cdots, a_n\}$ *is defined over* $\mathbb{Q}$ *and* $a_i \in (0,1)$, *then for any* $p \nmid lcd(\alpha)$, α *is closed under the p-adic Dwork dash ′.*
- *When* $\beta \neq \{1, 1, \cdots, 1\}$, *min of the step function may be negative. More adjustment will be necessary. It was considered by Long.*

Example



$$p \cdot {}_6F_5\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & 1 & 1 & 1 & \frac{7}{6} & \frac{5}{6} \end{matrix} ; \lambda\right]_{p-1} \equiv p \cdot H_p(\alpha, \beta; \lambda) \mod p.$$

$$pH_p(\alpha,\beta;1) = a_p(f_{4.6.a.a}) + p\cdot a_p(f_{12.4.a.a}) + \left(\frac{3}{p}\right)p^2$$

$$p\cdot {}_6F_5\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & \frac{5}{6} & \frac{7}{6} & 1 & 1 & 1\end{matrix};\ 1\right]_{p-1} \overset{?}{\equiv} a_p(f_{4.6.a.a}) \mod p^5.$$

$$\frac{1}{\pi}\,{}_6F_5\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & \frac{5}{6} & \frac{7}{6} & 1 & 1 & 1\end{matrix};\ 1\right]$$
$$= 6i\oint_{|t_3|=1}\left(\frac{1}{3}+\tau+\tau^2\right)\cdot\left(f_{4.6.a.a}(\tau/2) - 27f_{4.6.a.a}(3\tau/2)\right)d\tau,$$

where

$$t_3(\tau) = 4\left(\frac{1}{3\sqrt{3}}\frac{\eta^6(\tau)}{\eta^6(3\tau)} + 3\sqrt{3}\frac{\eta^6(3\tau)}{\eta^6(\tau)}\right)^{-2}$$

Some origins of "supercongruences"

- Beukers studied Apéry numbers for the proofs of $\zeta(2), \zeta(3) \notin \mathbb{Q}$. One of them

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = {}_4F_3 \left[\begin{matrix} -n & -n & n+1 & n+1 \\ & 1 & 1 & 1 \end{matrix} ; 1 \right]$$

He showed that for $m, r \geq 1, p > 3$,

$$u_{mp^r-1} \equiv u_{mp^{r-1}-1} \mod p^{3r}.$$

He conjectured that for each prime $p > 3$, modulo p^2

$$u_{\frac{p-1}{2}} = {}_4F_3 \left[\begin{matrix} \frac{1-p}{2} & \frac{1-p}{2} & \frac{1+p}{2} & \frac{1+p}{2} \\ & 1 & 1 & 1 \end{matrix} ; 1 \right] \equiv a_p(\eta(2\tau)^4\eta(4\tau)^4).$$

It was proved by Ahlgren and Ono using Greene version of finite hypergeometric functions, inspired Kilbourn's work.

Some origins of "supercongruences"

- Ramanujan-type formulas for $1/\pi$, one of them

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3}{(n!)^3}(6n+1)\frac{1}{4^n} = {}_4F_3\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{7}{6} \\ & 1 & 1 & \frac{1}{6} \end{matrix} ; \frac{1}{4}\right].$$

- Van Hamme made a few conjectures. E.g., for primes $p > 3$,

$${}_4F_3\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{7}{6} \\ & 1 & 1 & \frac{1}{6} \end{matrix} ; \frac{1}{4}\right]_{p-1} = \left(\frac{-1}{p}\right) p \mod p^4.$$

- More results and conjectures from Z.W. Sun.
- Inspired by McCarthy-Osburn and Zudilin, Long proved it using a p-adic perturbation method applied to a formula of Gessel and Stanton. This method was expanded in a paper by Long and Ramakrishna, adopted by Swisher to establish most of Van Hamme conjectures and by other researchers.

Some origins of "supercongruences"

- Rodriguez-Villegas made a few supercongruences conjectures regarding hypergeometric Calabi-Yau manifolds. (The ${}_2F_1(1)$ and ${}_3F_2(1)$ cases are proved by Mortenson.) In particular, for each $\alpha = \{r_1, 1-r_1, r_2, 1-r_2\}$ such that $r_i \in (0,1)$ and α is defined over $\mathbb{Q}$, there exists a weight 4 modular form f_α satisfying all primes $p > 5$

$$ {}_4F_3\left[\begin{matrix} r_1 & r_2 & 1-r_1 & 1-r_2 \\ & 1 & 1 & 1 \end{matrix} ; 1\right]_{p-1} \equiv a_p(f_{\alpha}) \mod p^3. $$

  Results by Kilbourn, McCarthy and Fuselier-McCarthy. This conjecture was proved by Long, Tu, Yui and Zudilin.
- Roberts and Rodriguez-Villegas made new supercongruence conjectures in 2017.

The p-adic perturbation method

Goal: Truncated HGS $(\star) \equiv R(\diamond) \mod p^n$

1. Understand the nature of R. Namely is it $\pm p, H_p, \gamma_p, a_p(f)$?
   $(\diamond = \clubsuit?, \heartsuit?, \spadesuit?, \diamondsuit?$, say $\diamond = \clubsuit)$
2. Find a formula for R (put $\clubsuit$ into its background)

$$\begin{array}{ccc} & \clubsuit & \\ \clubsuit & \clubsuit & \clubsuit \end{array}$$

3. Deform if necessary to peel off the desired truncated sum $(\star)$ as the major term and organize error terms in layers if possible

$$\begin{array}{ccc} & \clubsuit & \\ \clubsuit & \star & \clubsuit \end{array}$$

4. Eliminate the error terms $\odot$

$$\begin{array}{ccc} & \clubsuit & \\ \clubsuit & \star & \clubsuit \end{array}$$

The p-adic perturbation method in action

Van Hamme conjectured any odd prime p

$$\sum_{k=0}^{\frac{p-1}{2}}(4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3(-1)^k \equiv (-1)^{\frac{p-1}{2}}p \mod p^3.$$

It was first proved by Mortenson, another proof was by Zudilin using the Wilf-Zeilberger (WZ) method.

Now we look for a formula

$$\begin{array}{ccccc}
\partial & & & & \\
 & \partial & & \clubsuit & \\
 & \partial & \clubsuit & \clubsuit & \clubsuit \\
 & \partial & \partial & \partial &
\end{array}$$

The p-adic perturbation method in action

Goal:

$$\sum_{k=0}^{\frac{p-1}{2}}(4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3(-1)^k \equiv (-1)^{\frac{p-1}{2}}p \mod p^3.$$

A Whipple formula says

$$ {}_4F_3\left[\begin{matrix} a, & 1+a/2, & c, & d; & -1 \\ & a/2, & 1+a-c, & 1+a-d & \end{matrix}\right] = \frac{\Gamma(1+a-c)\Gamma(1+a-d)}{\Gamma(1+a)\Gamma(1+a-c-d)}$$

Letting $a=\frac{1}{2}, c=\frac{1}{2}+\frac{p}{2}, d=\frac{1}{2}-\frac{p}{2}$, the right hand side is $(-1)^{\frac{p-1}{2}}p$ (Found one $\clubsuit$ and its background).

The left hand side becomes

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1+p}{2} & \frac{1-p}{2} \\ & \frac{1}{4} & 1-\frac{p}{2} & 1+\frac{p}{2} \end{matrix}\, ;\ -1\right]
&= \sum_{k=0}^{\frac{p-1}{2}} (1+4k)\frac{(\frac{1}{2})_k(\frac{1-p}{2})_k(\frac{1+p}{2})_k}{k!(1-\frac{p}{2})_k(1+\frac{p}{2})_k}(-1)^k \\
&\equiv {}_4F_3\left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{4} & 1 & 1 \end{matrix}\, ;\ -1\right]_{p-1} \quad \bmod p^2.
\end{aligned}
$$

$$\clubsuit = \star = \star$$

To achieve the congruence modulo p^3, we consider

$$
{}_4F_3\left[\begin{matrix} \frac{1-p}{2}, & \frac{5}{4}, & \frac{1-x}{2}, & \frac{1+x}{2}; & -1 \\ & \frac{1}{4}, & 1+\frac{x}{2}, & 1-\frac{x}{2} & \end{matrix}\right]
= \sum_{k=0}^{\frac{p-1}{2}} (4k+1)\left(\frac{(\frac{1}{2})_k}{k!}\right)^3 (-1)^k + A_2x^2 + A_3x^4 + \cdots ,
$$

as a function of $x \in \mathbb{Z}[1/2][[x]]$, it is even. The goal next is to show $p \mid A_2$.

Now we look for another formula again.

$$
\begin{array}{ccccc}
\partial & & & & \\
 & \partial & & \clubsuit & \\
 & \partial & \clubsuit & \clubsuit & \clubsuit \\
 & \partial & \partial & \partial &
\end{array}
$$

Use another formula of Whipple.

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & c, & d, & e; & -1 \\ & \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e & \end{matrix}\right] \\
&= \frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)} \times \\
&{}_3F_2\left[\begin{matrix} 1+a-b-c, & d, & e; & 1 \\ & 1+a-b, & 1+a-c & \end{matrix}\right].
\end{aligned}
$$

Letting $a=\frac{1}{2}, b=\frac{1-x}{2}, c=\frac{1+x}{2}, e=\frac{1-p}{2}, d=1$, we have

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix} \frac{1}{2}, & \frac{5}{4}, & \frac{1-x}{2}, & \frac{1+x}{2}, & \frac{1-p}{2}, & 1; & -1 \\ & \frac{1}{4}, & 1+\frac{x}{2}, & 1-\frac{x}{2}, & \frac{1}{2}, & 1+\frac{p}{2} & \end{matrix}\right] \\
&\quad = \frac{\Gamma(\frac{1}{2})\Gamma(1+\frac{p}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{p}{2})}\,{}_3F_2\left[\begin{matrix} \frac{1}{2}, & 1, & \frac{1}{2}-\frac{p}{2}; & 1 \\ & 1+\frac{x}{2}, & 1-\frac{x}{2} & \end{matrix}\right].
\end{aligned}
\tag{5}
$$

Since $\dfrac{\Gamma(\frac{1}{2})\Gamma(1+\frac{p}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{p}{2})} = p$, every x-coefficient of the above is in $p\mathbb{Z}_p$. From which we conclude, $p \mid A_2$.

Remarks

- It works pretty well if the right hand side is either a character times a p-power or is an algebraic number which can be written as p-adic Gamma values (such as CM periods). E.g. Long and Ramakrishna showed the following conjectured by Kibelbek. For any prime $p \equiv 1 \mod 4$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ & 1 & 1 \end{matrix}; -\frac{1}{8}\right]_{p-1} \equiv -\left(\frac{-2}{p}\right)\Gamma_p\left(\frac{1}{4}\right)^4 \mod p^3. $$

- Meanwhile, it replies on existing identities, which makes it harder when the right hand side is more general.
- New identifies are found using the Wilf-Zeilberger (WZ) method.
- We will mention an easier way using a reside-sum technique.

Theorem (Long, Tu, Yui and Zudilin)

Let $HD = \{\alpha = \{r_1, r_2, 1-r_1, 1-r_2\}, \beta = \{1,1,1,1\}; \lambda = 1\}$, *where* $r_1, r_2 \in \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$ *or* $(r_1, r_2) = (\frac{1}{5}, \frac{2}{5}), (\frac{1}{10}, \frac{3}{10}), (\frac{1}{8}, \frac{3}{8})$, $(\frac{1}{12}, \frac{5}{12})$. *For each case there exists a weight-4 Hecke cuspidal eigenform* f_α *such that for any prime* $p > 5$ *be a prime. :*

$$H_p(\boldsymbol{\alpha}, \boldsymbol{\beta}; 1) = a_p(f_{\boldsymbol{\alpha}}) + \chi_{\boldsymbol{\alpha}}(p) \cdot p, \tag{6}$$

$$ {}_4F_3\left[\begin{matrix} r_1 & r_2 & 1-r_1 & 1-r_2 \\ & 1 & 1 & 1 \end{matrix} ; \ 1\right]_{p-1} \equiv a_p(f_{\boldsymbol{\alpha}}) \mod p^3. \tag{7}$$

There are two ways to look at the rigt hand side.

- If p is ordinary, it can be replaced by Dwork unit root
- In general, the H_p formula (6) plus the Gross-Koblitz formula will work as long as the error terms can be eliminated.

From Dwork to supercongruence

Lemma

Let $k \in \mathbb{Z}_{\geq 0}$, $a = [k]_0$ and $b = (k-a)/p$, that is, $k = a + bp$. Then for any $r \in \mathbb{Z}_p^\times$

$$\frac{(r)_k}{(1)_k} = \frac{-\Gamma_p(r+k)}{\Gamma_p(1+k)\,\Gamma_p(r)} \frac{(r')_b}{(1)_b} \cdot \left((r'+b)p\right)^{\nu(a,[-r]_0)},$$

where

$$\nu(a,x) = -\left\lfloor \frac{x-a}{p-1} \right\rfloor = \begin{cases} 0 & \text{if } \ a \leq x, \\ 1 & \text{if } x < a < p. \end{cases}$$

Assume $[k]_0 = a$, i.e. $k = a + bp$ where $a \in [0, p-1]$. Then we get a key reduction formula

$$\frac{(r)_{a+bp}}{(1)_{a+bp}} = \frac{(r)_a}{a!} \frac{(r')_b}{(1)_b} \left(1 + \frac{b}{r'}\right)^{\nu(a,[-r]_0)} \frac{\Gamma_p((r+a)+bp)\Gamma_p(1+a)}{\Gamma_p(r+a)\Gamma_p((1+a)+bp)}.$$

(8)

Theorem (Long, Tu, Yui, and Zudilin)

Let $\alpha = \{r_1, r_2, 1-r_1, 1-r_2\}$ be one of the fourteen multi-sets and p a prime such that $r_1, r_2 \in \mathbb{Z}_p^\times$. Let $F_s(\alpha) := F(\alpha, \{1,1,1,1\}; 1)_{p^s-1}$. Then for any integer $s \geq 1$,

$$F_{s+1}(\alpha) \equiv F_s(\alpha) F_1(\alpha) \mod p^3.$$

Idea:

$$\begin{aligned} F_{s+1}(\alpha) &= \sum_{a=0}^{p-1} \sum_{b=0}^{p^s-1} \frac{\prod_{j=1}^4 (r_j)_{a+bp}}{(1)_{a+bp}^4} \\ &= \sum_{b=0}^{p^s-1} \frac{\prod_{j=1}^4 (r_j')_b}{b!^4} \sum_{a=0}^{p-1} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4} \\ &\qquad \times \Lambda_\alpha(a+bp) \frac{\prod_{j=1}^4 \Gamma_p((r_j+a)+bp)}{\Gamma_p((1+a)+bp)^4}. \end{aligned} \tag{9}$$

where $\Lambda_{\boldsymbol{\alpha}}(a+bp) := \prod_{j=1}^4 \left(1 + \frac{b}{r_j'}\right)^{\nu(a,[-r_j]_0)}$

Putting together

$$
\begin{aligned}
F_{s+1}(\boldsymbol{\alpha}) &= \sum_{a=0}^{p-1} \sum_{b=0}^{p^s-1} \frac{\prod_{j=1}^4 (r_j)_{a+bp}}{(1)_{a+bp}^4} \\
&\equiv \sum_{b=0}^{p^s-1} \frac{\prod_{j=1}^4 (r_j')_b}{b!^4} \sum_{a=0}^{p-1} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4} \\
&\qquad \times \Lambda_{\boldsymbol{\alpha}}(a+bp)\big(1 + J_1(a)\cdot bp + J_2(a)\cdot (bp)^2\big) \mod p^3.
\end{aligned}
$$

$$
J_1(a) = J_1(a,\boldsymbol{\alpha}) := \sum_{j=1}^4 \big(G_1(r_j+a) - G_1(1+a)\big),
$$

$$
\begin{aligned}
J_2(a) = J_2(a,\boldsymbol{\alpha}) &:= 10\, G_1(1+a)^2 - 4\, G_1(1+a) \sum_{j=1}^4 G_1(r_j+a) \\
&+ \sum_{1\le j<\ell\le 4} G_1(r_j+a) G_1(r_\ell+a) + \frac{1}{2}\sum_{j=1}^4 \big(G_2(r_j+a) - G_2(1+a)\big).
\end{aligned}
$$

The error terms

$$C_1 = p\sum_{a=0}^{a_1} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4} J_1(a) + \sum_{a=a_1+1}^{a_2} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4}\left(\frac{1}{r_1'} + pJ_1(a)\right)$$
$$+ \left(\frac{1}{r_1'} + \frac{1}{r_2'}\right) \sum_{a=a_2+1}^{a_3} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4};$$
$$C_2 = p^2\sum_{a=0}^{a_1} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4} J_2(a) + \frac{p}{r_1'} \sum_{a=a_1+1}^{a_2} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4} J_1(a)$$
$$+ \frac{1}{r_1' r_2'} \sum_{a=a_2+1}^{a_3} \frac{\prod_{j=1}^4 (r_j)_a}{a!^4},$$

We would like to show they are zero modulo p^3.

An identity from the residue-sum method

For

$$R(t) = \frac{\prod_{i=1}^{n}(t-i)^2}{\prod_{i=0}^{n}(t+i)^2} = \sum_{k=0}^{n} \frac{B_k}{t+k} + \sum_{k=0}^{n} \frac{A_k}{(t+k)^2},$$

$$A_k = R(t)(t+i)^2|_{t=-k} = \frac{(k+1)_n^2}{k!^2(n-k)^2} = \binom{n}{k}^2 \binom{n+k}{k}^2.$$

$$B_k = \frac{d\left(R(t)(t+k)^2\right)}{dt}|_{t=-k} = A_k\left(-2H_{n+k} - 2H_{n-k} + 4H_k\right).$$

$$\sum_{k=0}^{n} B_k = \sum_{k=0}^{n} Res_{t=-k} R(t) = -Res_{t=\infty} R(t) = 0$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(H_{n+k} + H_{n-k} - 2H_k\right) = 0. \qquad (10)$$

The identity (10) implies when $(r_1, r_2) = (\frac{1}{2}, \frac{1}{2})$, $C_1 \equiv 0 \mod p^2$.

Here is a refinement, for each case, the modified rational function in action is

$$R(t) = \frac{\prod_{j=1}^{4} \prod_{i=1}^{a_j} (t - i + pr'_j)}{\prod_{i=0}^{p-1} (t+i)^2}, \quad a_j = [-r_j]_0.$$

The corresponding identity implies

$$C_1 \equiv C_2 \equiv 0 \mod p^3.$$

Second approach using character sums, which works for almost all primes p

It has been done using modularity plus the Gross-Koblitz formula. It turns out the error terms are more involved. We will need identities like

$$\sum_{k=1}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 (1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k) = 0, \quad (11)$$

which has already been already discovered via WZ, and used by Ahlgren and Ono to prove Beukers' conjecture.
It turns out for each (r_1, r_2) in the list,

$$tR(t) = t\frac{\prod_{j=1}^{4}\prod_{i=1}^{a_j}(t - i + pr_j')}{\prod_{i=0}^{p-1}(t+i)^2}, \quad a_j = [-r_j]_0$$

is sufficient to eliminate the two graded error terms modulo p^3.

Conjecture (Roberts and Rodriguez-Villegas)

Let $\alpha=\{a_1,\cdots,a_n\}$, $\beta=\{1,\cdots,1\}$ *be multi-sets satisfying defined over* $\mathbb{Q}$ *and* $a_i\in(0,1)$, $\lambda=\pm1$. *Let* A *be the unique submotive of the hypergeometric motive corresponding to* $\{\alpha,\beta;\lambda\}$ *with hodge number* $h^{0,n-1}(A)=1$ *and* r *the smallest positive integer such that* $h^{r,n-1-r}(A)=1$. *For any* $p\nmid lcd(\alpha,\beta)$ *and ordinary for* $\{\alpha,\beta;\lambda\}$, *there is a* p*-adic unit* $\mu_{\alpha,\beta;\lambda,p}$ *depending on the hypergeometric datum such that for any integer* $s\geq1$

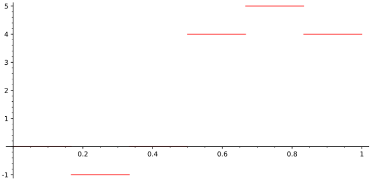
$$F(\alpha,\beta;\lambda)_{p^s-1}/F(\alpha,\beta;\lambda)_{p^{s-1}-1}\equiv\mu_{\alpha,\beta;\lambda,p}\mod p^{rs}.$$

Question

How to find such kinds of special hypergeometric data?

One recent approach by Li, Long and Tu was to use a Whipple ${}_7F_6(1)$-formula and its finite field analogue, as mentioned in Lecture III.

$$\alpha = \{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}\},\ \beta = \{1, 1, 1, 1, \tfrac{7}{6}, \tfrac{5}{6}\}$$



$$pH_p(\alpha, \beta; 1) = a_p(f_{4.6.a.a}) + p \cdot a_p(f_{12.4.a.a}) + \left(\frac{3}{p}\right) p^2$$

$$p \cdot {}_6F_5 \left[\begin{array}{cccccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & 1 & 1 & 1 & \frac{7}{6} & \frac{5}{6} \end{array}; \ \lambda \right]_{p-1} \equiv p \cdot H_p(\alpha, \beta; \lambda) \mod p.$$

$$p \cdot {}_6F_5 \left[\begin{array}{cccccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ & \frac{5}{6} & \frac{7}{6} & 1 & 1 & 1 \end{array}; \ 1 \right]_{p-1} \overset{?}{\equiv} a_p(f_{4.6.a.a}) \mod p^5.$$

Take-aways

- Hypergeometric functions have a lot of symmetries.
- There are compatible hypergeometric perspectives which shed lights to each other.
- They are explicit and can be explored theoretically or computationally.

Thank you!