Math 5020

Galois Representations $\circlearrowright G_{\mathbb{Q}}$

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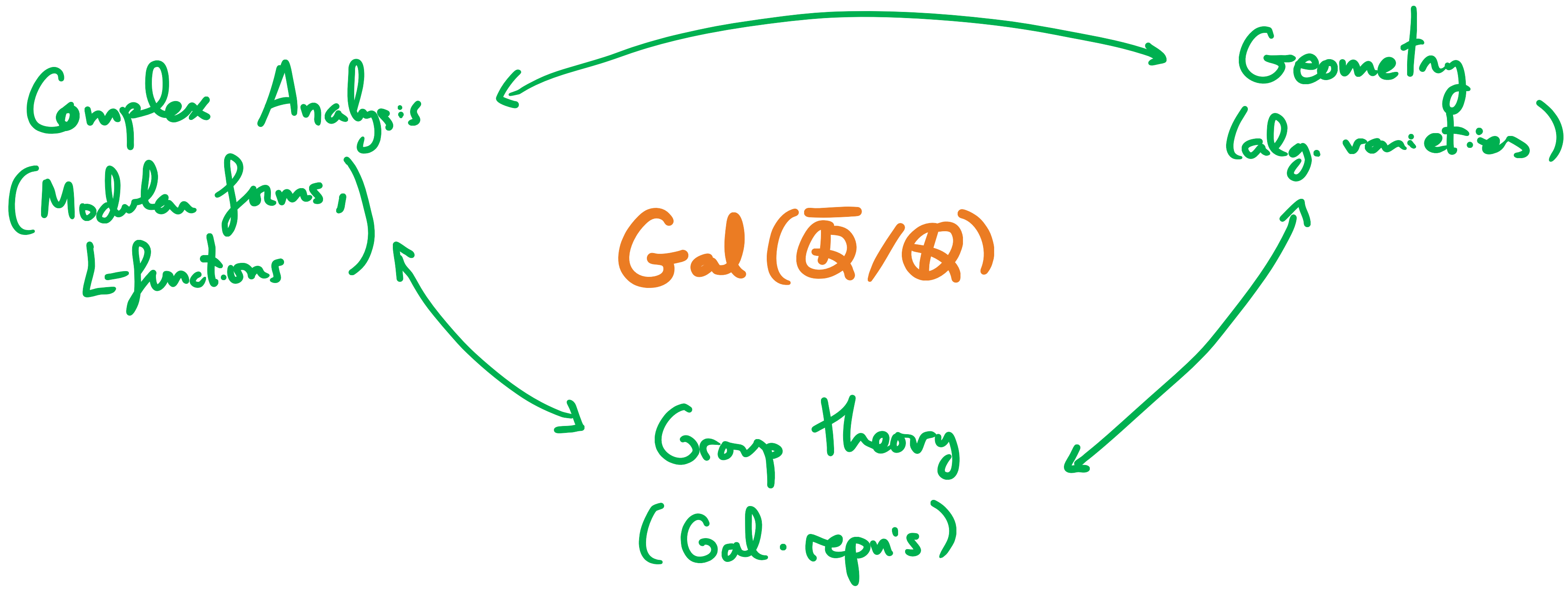
Goal Study $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) =: G_{\mathbb{Q}}$ = field automorphisms of $\overline{\mathbb{Q}}$ (over $\mathbb{Q}$)

$(\mathrm{Gal}(F/F))$

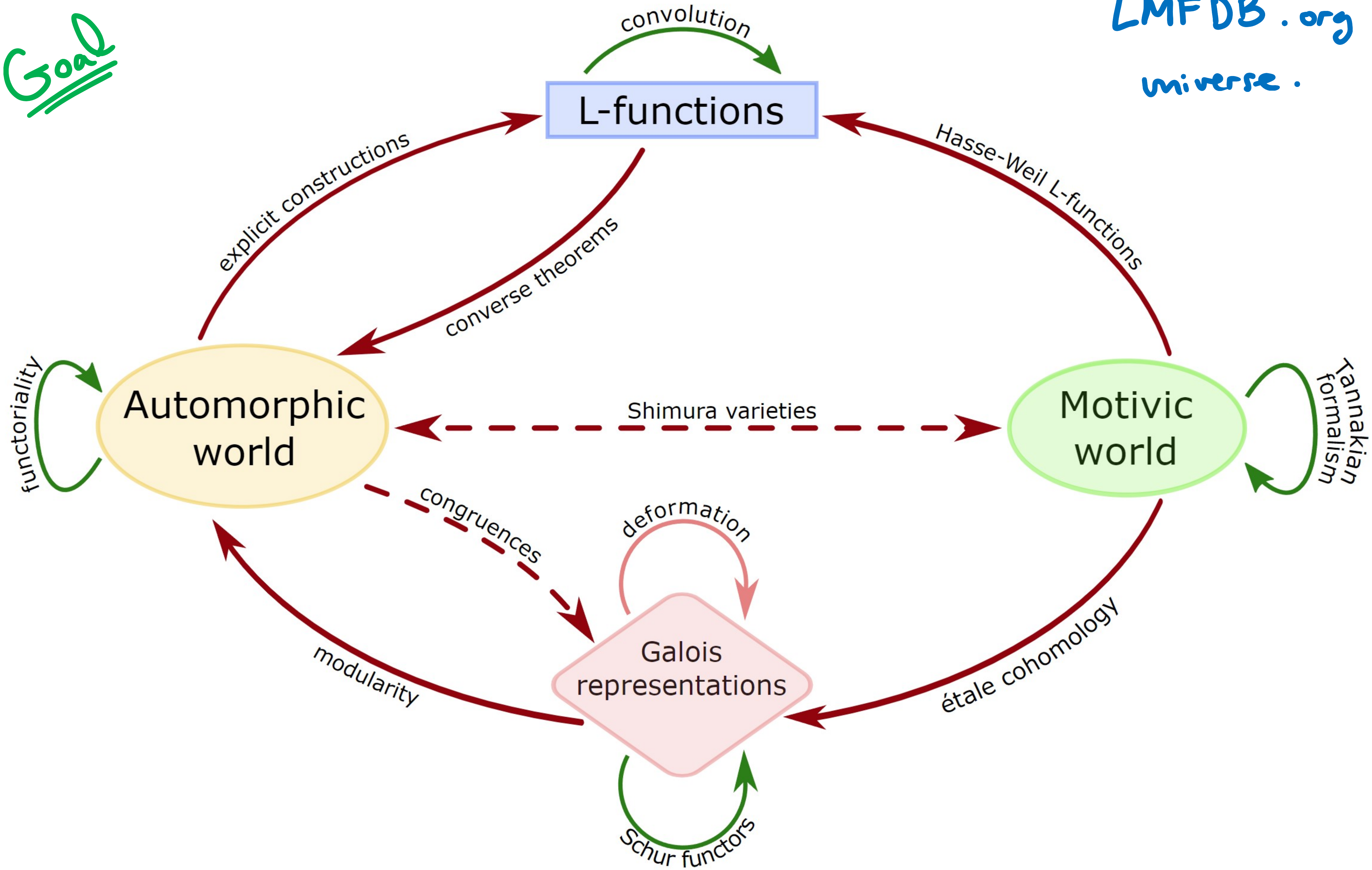
How? Thru its representations $\rho : G_{\mathbb{Q}} \longrightarrow H$ ← grp.



Goal

LMFDB.org

universe.



"Complex Analysis"

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \dots$$

Dirichlet L-function

(entire in $\mathbb{C}$, Euler prod., fn. eq'n)

"Geometry"

$$x^2 + 3 = 0$$

$$(x + \sqrt{-3})(x - \sqrt{-3}) = 0$$

"Algebra"

$$\mathbb{Q}(\sqrt{-3})$$

$$\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mu_2 = \{\pm 1\}$$

$$\sigma \longmapsto \frac{\sigma(\sqrt{-3})}{\sqrt{-3}} = \begin{cases} 1 & \text{if } \sigma(\sqrt{-3}) = \sqrt{-3} \\ -1 & \text{if } \sigma(\sqrt{-3}) = -\sqrt{-3} \end{cases}$$

χ is a Gal representation.

$$\chi : \mathrm{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q}) \cong (\mathbb{Z}/3\mathbb{Z})^{\times} \longrightarrow \mu_2$$

$$\chi : \mathbb{Z}/3\mathbb{Z} \to \{\pm 1\} \cup \{0\}$$

$$1 \mapsto 1$$

$$2 \mapsto -1$$

$$0 \mapsto 0$$

"Complex Analysis" $\quad q = e^{2\pi i \tau}$ $\quad \xrightarrow{\text{MODULARITY}}$ "Geometry" $\quad$ CM

$$f(\tau) = \sum_{n\geq 1} a_n q^n = q - 4q^7 + 2q^{13} + 8q^{19} + \cdots$$

$$E : y^2 = x^3 + 1 \ / \mathbb{Q}$$

MOD. FORM.

$$L(E,s) = \sum_{n\geq 1} \frac{a_n}{n^s} = 1 - \frac{4}{7^s} + \frac{2}{13^s} + \frac{8}{19^s} + \cdots$$

Hasse-Weil

$$G_{\mathbb{Q}}$$

$$E(\overline{\mathbb{Q}}) \circlearrowleft G_{\mathbb{Q}}$$

$$\underbrace{E(\overline{\mathbb{Q}})[p]}_{p\text{-torsion}} \circlearrowleft G_{\mathbb{Q}}$$

"Algebra"

$$\rho_E : G_{\mathbb{Q}} \to \mathrm{Aut}(T_p(E))$$

p-adic Galois representation

$$\mathrm{Aut}(T_p(E)) \to \mathrm{Aut}(E[p^n]) \to \mathrm{Aut}(E[p])$$

$$G_{\mathbb{Q}} \to \mathrm{Aut}(E[p])$$

$$\sigma \mapsto (P \mapsto \sigma(P))$$

What is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$?

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \varprojlim_{\substack{F/\mathbb{Q} \\ \text{fin. Galois}}} \mathrm{Gal}(F/\mathbb{Q})$$

- $\mathbb{Q}(2^\infty) = \mathbb{Q}(\{\sqrt{d} : d \in \mathbb{Z}\}) \subsetneq \overline{\mathbb{Q}}$

$$\mathrm{Gal}(\mathbb{Q}(2^\infty)/\mathbb{Q}) = \varprojlim \mathrm{Gal}(\mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{p_n})/\mathbb{Q})$$
$$= \varprojlim (\mathbb{Z}/2\mathbb{Z})^{n+1} = \prod_{k=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$$

- $\mathbb{Q}(\mu_{p^\infty}) = \mathbb{Q}(\{\zeta_{p^n} : n \geq 1\}) \subsetneq \overline{\mathbb{Q}}$

$$\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) = \varprojlim \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \cong \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times$$
$$\cong \mathbb{Z}_p^\times$$

← the p-adic integers (units of)

- $\mathbb{Q}^{ab}$ = Compositum of <u>all</u> abelian extensions of $\mathbb{Q}$ $\subsetneq \overline{\mathbb{Q}}$

$$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) = \varprojlim_{\substack{F/\mathbb{Q} \\ \text{fin. Galois,} \\ \text{abelian}}} \mathrm{Gal}(F/\mathbb{Q}) = ?$$

$$= \varprojlim_{n} \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \varprojlim (\mathbb{Z}/n\mathbb{Z})^\times$$

Kronecker - Weber

$$= \left(\varprojlim \mathbb{Z}/n\mathbb{Z}\right)^\times$$

$$= \hat{\mathbb{Z}}^\times$$

$$= \prod_p \mathbb{Z}_p^\times$$

$\mathbb{Q}^{ab} = \prod \mathbb{Q}(\zeta_{p^\infty})$

- $E/\mathbb{Q}$ ell curve (e.g. $y^2 = x^3+1$) $\rightsquigarrow$ $E(\overline{\mathbb{Q}})[p^n] \circlearrowleft G_{\mathbb{Q}}$

$$G_{\mathbb{Q}} \circlearrowright \mathbb{Q}(E[p^n]) = \mathbb{Q}(\{x(P), y(P) : P \in E[p^n]\})$$

p^n- division field

$T_p(E)$ = p-adic Tate module of $E/\mathbb{Q}$

$$E[p^n] \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$$

$$\mathrm{Aut}(E[p^n]) \cong GL(2, \mathbb{Z}/p^n\mathbb{Z})$$

$$\mathbb{Q}(T_p(E)) = \bigcup_{n \geq 1} \mathbb{Q}(E[p^n])$$

$$\begin{aligned}
\mathrm{Gal}(\mathbb{Q}(T_p(E))/\mathbb{Q}) &= \varprojlim \mathrm{Gal}(\mathbb{Q}(E[p^n])/\mathbb{Q}) \\
&\subseteq \varprojlim GL(2, \mathbb{Z}/p^n\mathbb{Z}) \\
&= GL(2, \mathbb{Z}_p)
\end{aligned}$$

Note: • $\mathbb{F}$ is a finite field, then

$$\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \cong \widehat{\mathbb{Z}} \quad \left(= \varprojlim \mathbb{Z}/n\mathbb{Z}\right)$$

• $\mathbb{Q}_p$ is the field of p-adic numbers

$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ can also be described explicitly.

• $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \subseteq \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

as a decomposition gp.

What is a Gal. rep'n?

A gp. hom. $\rho : G_{\mathbb{Q}} \longrightarrow H$ where H is another gp

We typically consider linear Gal rep'ns:

$$\rho : G_{\mathbb{Q}} \longrightarrow GL_n(F)$$

where F is a field:

- $F = \mathbb{C}$ $\rightsquigarrow$ Artin Galois rep'ns.
- $F \subseteq \overline{\mathbb{F}_p}$ $\rightsquigarrow$ mod-p Galois rep'ns.
- $F \subseteq \overline{\mathbb{Q}_\ell}$ $\rightsquigarrow$ ℓ-adic Galois rep'ns.

BUT rep'ns NEED to be continuous!

$\rightsquigarrow$

TOPOLOGY