MATH 5020

GALOIS REPRESENTATIONS $\circlearrowright G_{\mathbb{Q}}$

LECTURE 2

SPRING 2022

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MONT 233

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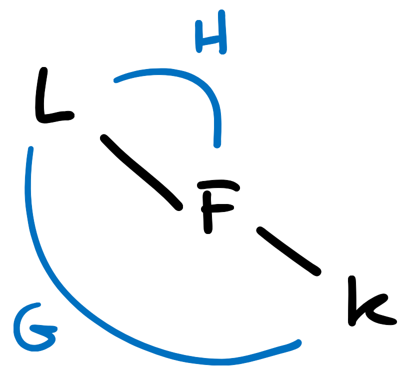
$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow H$ "TOPOLOGY"

In the finite case : L/K finite Galois, $\mathrm{Gal}(L/K) = G$

$$\{\text{subfields } K \subseteq F \subseteq L\} \longleftrightarrow \{\text{subgps } \{e\} \subseteq H \subseteq G\}$$

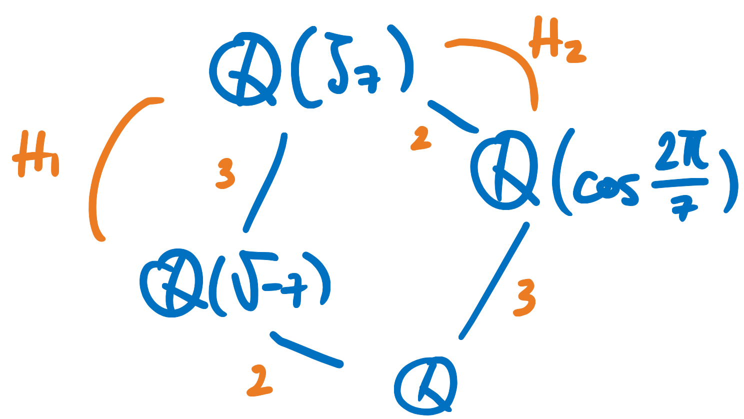
$$F \longmapsto \mathrm{Gal}(L/F)$$

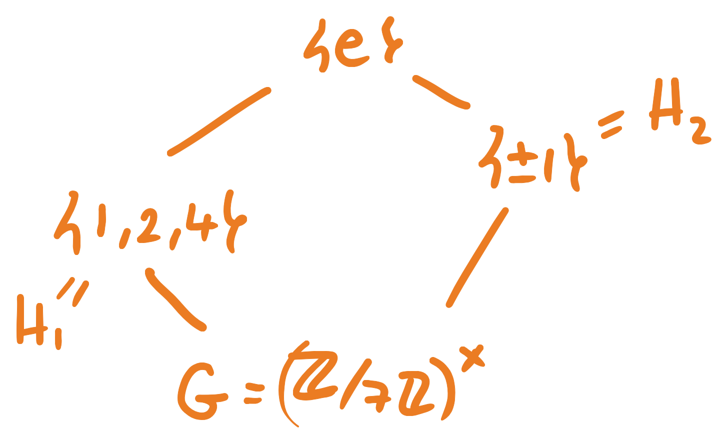
$$L^H \longleftarrow H$$



s.t. F/K is Galois $\iff H \triangleleft G$.

ex $\mathbb{Q}(\zeta_7)$, $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q}) \cong (\mathbb{Z}/7\mathbb{Z})^\times \cong \mathbb{Z}/6\mathbb{Z}$





In the infinite case:

ex $L = \mathbb{Q}(2^{\infty}) = \mathbb{Q}(\{\sqrt{d} : d \in \mathbb{Z}\})$ What is $\mathrm{Aut}_{\mathbb{Q}}(L)$?

$= \mathbb{Q}(\sqrt{-1}, \sqrt{p} : p \text{ prime})$

Minimal set of alg. generators of L: $\alpha_0 = \sqrt{-1}$, $\alpha_k = \sqrt{p_k}$ where p_k is the kth prime.

$\phi \in \mathrm{Aut}_{\mathbb{Q}}(L)$ is determined by $\phi(i) = \pm i$, $\phi(\alpha_k) = \pm \alpha_k$

so $\mathrm{Aut}_{\mathbb{Q}}(L) = \prod_{j \geq 0} \{\pm 1\} \cong \prod_{k \geq 0} \mathrm{Gal}(\mathbb{Q}(\alpha_k)/\mathbb{Q})$

<u>Correspondence</u>?

$$\{\text{subfields } \mathbb{Q} \subseteq F \subseteq \mathbb{Q}(2^\infty)\} \overset{??}{\longleftrightarrow} \{\text{subgrps of } \mathrm{Aut}_{\mathbb{Q}}(L)\}$$

$$\mathbb{Q} \longleftrightarrow \mathrm{Aut}_{\mathbb{Q}}(L)$$

$$L \longleftrightarrow \{e\} = \{(1,1,\dots,1,\dots)\}$$

$$\mathbb{Q}(i) \longleftrightarrow \underbrace{\{1\}}_{\text{0-th coord}} \times \prod_{j\geq 1} \{\pm 1\}$$

$$\mathbb{Q}(\alpha_k) \longleftrightarrow \prod_{j=0}^{k-1} \{\pm 1\} \times \underbrace{\{1\}}_{\text{k-th position}} \times \prod_{j\geq k+1} \{\pm 1\}$$

$$\mathbb{Q}(\sqrt{6}) \longleftrightarrow \underbrace{\{\pm 1\}}_{0} \times \underbrace{\{(1,1),(-1,-1)\}}_{1+2} \times \underbrace{\prod \{\pm 1\}}_{\geq 3}$$

NOTE: L contains <u>all</u> quad. extensions of $\mathbb{Q}$ and there are countably many such extensions.

BUT $H \subseteq \prod_{k \geq 0} \{\pm 1\}$ of index 2 are given by kernels of surjective maps down to $\mathbb{F}_2$

$$\mathrm{Hom}\left(\prod \{\pm 1\}, \{\pm 1\}\right) = \underbrace{\mathrm{Hom}\left(\mathbb{F}_2^{\infty}, \mathbb{F}_2\right)}_{\text{this is uncountably infinite!}}$$

$\Rightarrow$ the Galois correspondence is impossible as it is!

SOLUTION!? $\longrightarrow$ <u>TOPOLOGY</u>

WOLFGANG
Krull (1899 - 1971)

View $\prod_{k \geq 0} \mathrm{Gal}(\mathbb{Q}(\alpha_k)/\mathbb{Q})$

as a topological space (top. gp.)

G w/ a topology s.t.:

- mult, inversion are continuous
- each factor has discrete topology

and the prod. space w/ product topology

$\rightsquigarrow$ coarsest topology for which all projections (fewest open sets)

$$p_k : \mathrm{Gal}(L/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathbb{Q}(\alpha_k)/\mathbb{Q})$$

are continuous.

$p_k^{-1}(\{id\})$ is open, that is $\mathrm{Gal}(L/\mathbb{Q}(\alpha_k))$ is open.

$p_k^{-1}(\{id, \tau_k\}) = \mathrm{Gal}(L/\mathbb{Q})$ is open.

$p_k^{-1}(\{\tau_k\})$ is open but not a subgp.

$\hookrightarrow$ $\mathrm{Gal}(L/\mathbb{Q}(\alpha_k))$ is closed.

Galois correspondence in the infinite case

$$\{\text{subfields } \mathbb{Q} \subseteq F \subseteq L\} \longleftrightarrow \{\text{closed subgp of } G\}$$

$$G = \mathrm{Aut}_{\mathbb{Q}}(L)$$

$$\{\text{subfields } \underbrace{\mathbb{Q} \subseteq F}_{\text{finite}} \subseteq L\} \longleftrightarrow \{\text{open subgps of } G\}$$

$\underline{Q}$: $L = \mathbb{Q}(2^{\infty})$, are there countably many open subgps of $\mathrm{Aut}_{\mathbb{Q}}(L)$ of index 2??

In prod topology $\mathrm{Gal}(L/\mathbb{Q}) \cong \prod_{k \geq 0} \mathrm{Gal}(\mathbb{Q}(\alpha_k)/\mathbb{Q})$

the open sets $p_k^{-1}(\{id_k\})$, $p_k^{-1}(\{\tau_k\})$ form a subbasis of the top.

$$p_k^{-1}(\{id_k\}) = \prod_{j \neq k} \{\pm 1\} \times \{id_k\}$$

$$\boxed{\text{The open sets } F_S = \prod_{k \notin S} \mathrm{Gal}(\mathbb{Q}(\alpha_k)/\mathbb{Q}) \times \mathcal{U}_S \qquad S \text{ is finite}, \ \mathcal{U}_S \subseteq \prod_{k \in S} \mathrm{Gal}}$$

form a basis of the topology

Let H be a closed subgp of finite index $\Longrightarrow$ H is open

$$\left(G = \underbrace{H \cup g_1 H \cup \dots \cup g_r H}_{\text{closed}} \Rightarrow H \text{ is open} \right)$$

Let H be a closed subgp of finite index $\Longrightarrow$ H is open $\Longrightarrow$ $H = \bigcup_{i \in I} F_{s_i}$

$\left(G = H \cup \underbrace{g_1 H \cup \dots \cup g_r H}_{\text{closed}} \Rightarrow H \text{ is open} \right)$

H closed, G compact (Tychonoff's theorem) $\Longrightarrow$ $H = \bigcup_{i=1}^{n} F_{s_i}$

- Since the F_S's are countably many, and H is a finite union of F_S's

$\Longrightarrow$ only countably many H closed subgp of finite index

Krull.