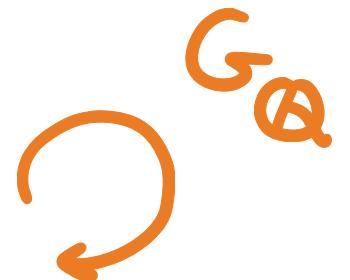


MATH 5020

GALOIS REPRESENTATIONS

LECTURE 2+ ε

SPRING 2022



Review of Galois Theory (Dummit + Foote, Ch 14)

Def Let K/F be an extension of fields.

$\text{Aut}(K/F) \cong \text{Aut}_F(K)$ is the collection of field automorphisms of K that fix F pointwise.

ex $\mathbb{Q}(i)/\mathbb{Q}$ $\phi: \mathbb{Q}(i) \rightarrow \mathbb{Q}(i)$ "complex conjugation"
 $a+bi \mapsto a-bi$

ex $\mathbb{F}_{p^2}/\mathbb{F}_p$ $\phi_p: \mathbb{F}_{p^2} \rightarrow \mathbb{F}_{p^2}$ "Frobenius automorphism"
 $k \mapsto k^p$

Prop K/F , $\alpha \in K$ is algebraic over F , $\sigma \in \text{Aut}(K/F)$

then $\sigma(\alpha)$ is a conjugate of α

(i.e., $\sigma(\alpha)$ is another root of the min. poly. of α over F .)

ex $\tau \in \text{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, $\zeta_n = e^{\frac{2\pi i}{n}}$, $x^n - 1 = 0$ $\xrightarrow{\text{not min'l}} \phi_n(x) = 0$

then $\tau(\zeta_n) = e^{\frac{2\pi i}{n} \cdot a}$ for some $1 \leq a \leq n$
 $\gcd(a, n) = 1$

... Thm $\text{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$

ex $\text{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{\text{id}\}$

b/c the only conjugate of $\sqrt[3]{2}$ in $\mathbb{Q}(\sqrt[3]{2})$ is itself!

Splitting field of $x^3 - 2$, $\text{Splt}(x^3 - 2) = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$
 $= \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$

$A = \text{Aut}(\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})/\mathbb{Q})$

$\phi_{i,j}: \sqrt{-3} \mapsto (-1)^i \sqrt{-3} \quad i=0,1$
 $\phi_{i,j}: \sqrt[3]{2} \mapsto \zeta_3^j \sqrt[3]{2} \quad j=0,1,2$

put $\tau = \phi_{1,0}$, $\sigma = \phi_{0,1} \Rightarrow \tau \circ \sigma = \sigma^2 \circ \tau \rightarrow A \cong S_3$.

Prop 1) $f(x) \in F[x]$, $E = \text{Splt}(f(x))$ then

$\# \text{Aut}(E/F) \leq [E:F]$ w/ equality if $f(x)$ is
separable over F
(no multiple roots)

2) Let K/F be any ext'n. Then $\# \text{Aut}(K/F) \leq [K:F]$
w/ equality if F is the fixed field of $\text{Aut}(K/F)$.

ex p prime, $\mathbb{F}_p(\tau)$, $f(x) = x^p - \tau = (x - \sqrt[p]{\tau})^p$

so $\text{Splt}(x^p - \tau) = \mathbb{F}_p(\tau)(\sqrt[p]{\tau}) = \mathbb{F}_p(\sqrt[p]{\tau})$ not separable

$$\begin{aligned} [\mathbb{F}_p(\sqrt[p]{\tau}) : \mathbb{F}_p(\tau)] &= p \\ \text{Aut}(\mathbb{F}_p(\sqrt[p]{\tau}) / \mathbb{F}_p(\tau)) &= \{\text{id}\} \end{aligned} \quad \left\{ \begin{array}{l} \text{so } 1 \neq P \\ \end{array} \right.$$

Def K/F finite. Then K is said to be Galois over F
and K/F is a Galois extension if

$$\#\text{Aut}(K/F) = [K : F]$$

In that case $\text{Gal}(K/F) = \text{Aut}(K/F)$.

