MATH 5020

GALOIS REPRESENTATIONS $\circlearrowright G_{\mathbb{Q}}$

LECTURE $2+\varepsilon$

SPRING 2022

Review of Galois Theory (Dummit + Foote, Ch 14)

Def Let K/F be an extension of fields.

$\mathrm{Aut}(K/F)$ or $\mathrm{Aut}_F(K)$ is the collection of field automorphisms of K that fix F pointwise.

ex $\mathbb{Q}(i)/\mathbb{Q}$ $\quad \phi : \mathbb{Q}(i) \longrightarrow \mathbb{Q}(i)$ $\quad$ "complex conjugation"

$\qquad a+bi \longmapsto a-bi$

ex $\mathbb{F}_{p^2}/\mathbb{F}_p$ $\quad \phi_p : \mathbb{F}_{p^2} \longrightarrow \mathbb{F}_{p^2}$ $\quad$ "Frobenius automorphism"

$\qquad k \longmapsto k^p$

Prop K/F, $\alpha \in K$ is algebraic over F, $\sigma \in \mathrm{Aut}(K/F)$

then $\sigma(\alpha)$ is a conjugate of α

(i.e., $\sigma(\alpha)$ is another root of the min. poly. of α over F.)

ex $\tau \in \mathrm{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, $\zeta_n = e^{\frac{2\pi i}{n}}$, $x^n - 1 = 0$ not min'l $\rightarrow \phi_n(x) = 0$

then $\tau(\zeta_n) = e^{\frac{2\pi i}{n} \cdot a}$ for some $1 \leq a \leq n$, $\gcd(a,n) = 1$

... **Thm** $\mathrm{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$

ex $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{id\}$

b/c the only conjugate of $\sqrt[3]{2}$ in $\mathbb{Q}(\sqrt[3]{2})$ is itself!

Splitting field of x^3-2, $\mathrm{Splt}(x^3-2) = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$

$= \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$

$A = \mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})/\mathbb{Q})$

$$\phi_{i,j} : \begin{array}{lll} \sqrt{-3} \longmapsto (-1)^i \sqrt{-3} & & i=0,1 \\ \sqrt[3]{2} \longmapsto \zeta_3^j \sqrt[3]{2} & & j=0,1,2 \end{array}$$

put $\tau = \phi_{1,0}$, $\sigma = \phi_{0,1}$ $\Rightarrow$ $\tau \cdot \sigma = \sigma^2 \circ \tau$ $\rightarrow$ $A \cong S_3$.

Prop 1) $f(x) \in F[x]$, $E = \text{Splt}(f(x))$ then

$\#\text{Aut}(E/F) \leq [E:F]$ w/ equality if $f(x)$ is separable over F (no multiple roots)

2) Let K/F be any ext'n. Then $\#\text{Aut}(K/F) \leq [K:F]$ w/ equality if F is the fixed field of $\text{Aut}(K/F)$.

ex p prime, $\mathbb{F}_p(T)$, $f(x) = x^p - T = (x - \sqrt[p]{T})^p$ ← not separable

so $\text{Splt}(x^p - T) = \mathbb{F}_p(T)(\sqrt[p]{T}) = \mathbb{F}_p(\sqrt[p]{T})$

$[\mathbb{F}_p(\sqrt[p]{T}) : \mathbb{F}_p(T)] = p$

$\text{Aut}(\mathbb{F}_p(\sqrt[p]{T}) / \mathbb{F}_p(T)) = \{id\}$

} so $1 \lneq p$

Def K/F finite. Then K is said to be Galois over F and K/F is a Galois extension if

$$\#\mathrm{Aut}(K/F) = [K:F]$$

In that case $\mathrm{Gal}(K/F) = \mathrm{Aut}(K/F)$.