Math 5020 - Galois Representations
Suggested Exercises

P1 Let $L = \mathbb{Q}(\sqrt{-1}, \sqrt{p} : p \text{ prime})$. Then, $A = \text{Aut}(L/\mathbb{Q}) = \prod_{k \geq 0} \{\pm 1\}$. Find a subgroup $H$ of $A$ of index 2 such that $L^H$, the subfield of $L$ fixed by $H$, cannot correspond to a quadratic extension of $\mathbb{Q}$ via the Galois correspondence.

P2 Prove that $\text{Hom}(\mathbb{F}_2^\infty, \mathbb{F}_2)$ is uncountably infinite.

P3 Let $F = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}(\sqrt{2}).$ Show that $F/\mathbb{Q}$ and $K/F$ are Galois, but $K/\mathbb{Q}$ is not Galois.

P4 Let $\mathbb{F}$ be a finite field of size $q = p^n$, where $p$ is prime and $n \geq 1$, and let $\phi : \mathbb{F} \rightarrow \mathbb{F}$ such that $\phi(k) = k^p$ be the Frobenius map. Show that $\phi$ is an automorphism of $\mathbb{F}$.

P5 Show that $F = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ is a Galois extension of $\mathbb{Q}$ and $\text{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$.

P6 Show that $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ is Galois, with $\text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

P7 Let $\pi \in \mathbb{C}$ be a transcendental number over $\mathbb{Q}$. Show that $\mathbb{Q}(\pi)/\mathbb{Q}$ contains infinitely many distinct intermediary subfields.

P8 Let $p$ be a prime and let $K = \mathbb{F}_p(x, y)$ and $F = \mathbb{F}_p[x^p, y^p]$. Let $L_n = \mathbb{F}_p[x^p, y^p, x + x^{p^n} y]$.

(a) Show that $F \subseteq L_n \subseteq K$ and all $L_n$ are distinct.

(b) Prove that $K/F$ is finite of degree $p^2$.

(c) Conclude that $K/F$ is a finite extension that contains infinitely many distinct intermediary subfields.

P9 (Dummit and Foote, Ch. 14.5, Problem 8)

P10 (Dummit and Foote, Ch. 14.7, Problem 19)

P11 Let $p > 2$ be a prime. Prove that the discriminant of the number field $\mathbb{Q}(\zeta_p)$ is $(-1)^{\frac{p-1}{2}} \cdot p^{p-2}$.

P12 Let $K/\mathbb{Q}$ be a finite extension of $\mathbb{Q}$.

(a) Prove that $K \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$ for all but finitely many prime numbers $p$.

(b) Suppose $K/\mathbb{Q}$ is Galois. Conclude that the cyclotomic character $\chi^K_p : \text{Gal}(K/\mathbb{Q}) \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times$ is trivial for all but finitely many prime numbers $p$.

P13 Let $K$ be a Galois extension of $\mathbb{Q}$ such that $\mathbb{Q}(\zeta_n) \subseteq K$ for some $n \geq 3$, and let $F \subseteq K$ be a subfield. Show that the restriction of the cyclotomic character $\chi_n^{K/F} : \text{Gal}(K/F) \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$ is surjective if and only if $\mathbb{Q}(\zeta_n) \cap F = \mathbb{Q}$.

P14 Let $\oplus$ be a binary operation defined on $\mathbb{Z}/p^n\mathbb{Z}$ by $k \oplus k' \equiv k + k' + kk'p \mod p^n$, for a fixed $n \geq 2$.

(a) Prove that $(\mathbb{Z}/p^n\mathbb{Z}, \oplus)$ is an abelian group.

(b) Show that $(\mathbb{Z}/p^n\mathbb{Z}, \oplus)$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z}, +)$ with its usual addition of congruences.

(c) Let $U_n$ be the subset of $(\mathbb{Z}/p^n\mathbb{Z})^\times$ formed by units that are congruent to 1 mod $p$. Show that $U_n$ is a subgroup and that $U_n \cong \mathbb{Z}/p^{n-1}\mathbb{Z}$, where $\mathbb{Z}/p^{n-1}\mathbb{Z}$ comes equipped with $\oplus$. 

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**P15** Let \( p \) be an odd prime and let \( K_p \) be the unique subfield of \( \mathbb{Q}(\zeta_p^2) \) such that \( [K_p : \mathbb{Q}] = p \).

For \( p = 3, 5, \) and \( 7 \), find a polynomial \( f_p(x) \) such that \( K_p \) is the splitting field of \( f_p(x) \). (You are welcome to use Magma to find these.)

**P16** Let \( p \) be a prime. Let \( G = \text{GL}(2, \mathbb{Z}_p) \) and \( G_n = \text{GL}(2, \mathbb{Z}/p^n\mathbb{Z}) \).

(a) Let \( \pi_n : \text{GL}(2, \mathbb{Z}_p) \to \text{GL}(2, \mathbb{Z}/p^n\mathbb{Z}) \) be the reduction mod-\( p^n \) map. Let \( H_n \) be a subgroup of \( \text{GL}(2, \mathbb{Z}/p^n\mathbb{Z}) \). Show that \( \pi_n^{-1}(H_n) \) is an open subgroup of \( \text{GL}(2, \mathbb{Z}_p) \) of finite index.

(b) Let \( H \) be a subgroup of \( \text{GL}(2, \mathbb{Z}_p) \) of finite index, and let \( H_n = \pi_n(H) \) for every \( n \geq 1 \).

Show that the natural map \( G/H \to G_n/H_n \) is surjective. Conclude that \( [G : H] \geq [G_n : H_n] \).

(c) With notation as in (b), show that there is some number \( N \geq 1 \) such that \( [G : H] = [G_n : H_n] \) for all \( n \geq N \).

(d) With notation as in (b) and (c), show that \( H = \pi_n^{-1}(H_N) \).