MATH 5020

Galois Representations

$G_{\mathbb{Q}}$

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§. Density

Def. Let X be a topological space and let $A \subseteq X$. We say A is dense in X if for every non-empty open $\mathcal{U} \subseteq X$ we have $A \cap U$ is also non-empty.

ex $\mathbb{R} \supseteq \mathbb{Q}$ $\quad (a,b) \subseteq \mathbb{R}$, $(a,b) \cap \mathbb{Q}$ infinite.

ex $\mathbb{Z} \subseteq \mathbb{Z}_p$ via $n \mapsto (n \bmod p,\ n \bmod p^2, \dots, n \bmod p^j, \dots)$

Claim $\mathbb{Z}$ is dense $\mathbb{Z}_p$.

Pf. Let U be an open set in $\mathbb{Z}_p$. We may assume that $\mathcal{U}$ is a ball $U = B(s,\delta)$, for some $s \in \mathbb{Z}_p$, $\delta > 0$.

Now $B(s,\delta) = \{ z \in \mathbb{Z}_p : |s-z|_p < \delta \}$

$= \{ z \in \mathbb{Z}_p : \frac{1}{p^{\nu_p(s-z)}} < \delta \}$

$= \{ z \in \mathbb{Z}_p : \nu_p(s-z) > n(\delta) \}$ where $n(\delta) = \lfloor \log_p 1/\delta \rfloor$

$= \{ z \in \mathbb{Z}_p : z \equiv s \mod p^{n(\delta)+1} \}$

$s = (s_1, s_2, \ldots, s_{n(\delta)+1}, \ldots)$

$\Rightarrow \underbrace{\{ m \in \mathbb{Z} : m \equiv s_{n(\delta)+1} \mod p^{n(\delta)+1} \}}_{\text{infinite!}} \subseteq B(s,\delta) \subseteq \mathcal{U}$

$\Rightarrow B(s,\delta) \cap \mathbb{Z} = \mathcal{U} \cap \mathbb{Z}$ is infinite ✓

ex $\mathbb{Z}_3 \ni \left(1+3+\cdots+3^{j-1} \bmod p^j\right)_{j\geq 1} = s$

$= (1, 1+3, 1+3+9, \ldots)$

$B(s, \frac{1}{4})$, $|z-s|_3 < \frac{1}{4} \Rightarrow z \equiv s \bmod 3^2$

$\Rightarrow 4 \in B(s, \frac{1}{4})$.

§. Products

Let $\{G_i\}_{i\in I}$ be a family of top. gps. Let $G = \prod G_i$ w/ natural gp str., prod topology. And let X be another top. gp.

Then a map $f: X \to G$ is continuous $\Longleftrightarrow$ $\pi_i \circ f : X \to G \to G_i$ is continuous.

$G = \prod G_i$, note $*$, $(\cdot)^{-1}$, are defined so diagrams:

$$\begin{tikzcd} G \times G \arrow[r, "*"] \arrow[d, "\pi_i \times \pi_i"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d] \\ G_i \times G_i \arrow[r, "*_i"] & G_i \end{tikzcd}$$

(similarly $(\cdot)^{-1}$)

$\Rightarrow$ π_i, $*_i$, $(\cdot)_i^{-1}$ are cont.

$\to$ $\pi_i \circ *_i$, $\pi_i \circ (\cdot)_i^{-1}$ are continuous

$\Rightarrow$ $*$, $(\cdot)^{-1}$ are cont. on G

$\to$ G is a top. gp. (in $\mathcal{TG}$)

In addition, H is a top. gp and we have morphisms $f_i : H \to G_i$

$\to$ the map $f : H \to G$ is a gp. morphism in $\mathcal{TG}$.
(both gp. hom, and cont.)

$\Rightarrow$ $G = \prod G_i$ is a product in $\mathcal{TG}$ as well.

ex $G = \prod G_i$, $H_i \lhd G_i$, $H = \prod H_i$

then $G/H \cong \prod G_i/H_i$ in TG.

ex $p>2$

$$\mathbb{Z}_p^\times/\mu_{p-1} \cong \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times / \mu_{p-1,n}$$

as TG.

§. Fundamental System of Neighborhoods

Def X top. space, $A \subseteq X$. An element $a \in A$ is in the interior of A if $\exists\, \mathcal{U} \subseteq X$ open s.t. $a \in \mathcal{U} \subseteq A$.

Def A neighborhood of $x \in X$ is a subset of X containing x as an interior point.

Def A fundamental system of neighborhoods (FSON) of $x \in X$ is a collection $\mathcal{F}$ of neighborhoods of x s.t. every neighborhood of x contains at least one member of $\mathcal{F}$ (likewise, open fund. systems, ...)

ex $\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a FSON of $0 \in \mathbb{R}$.

ex $[0, \varepsilon)$ is NOT a neighborhood of 0.

ex 0 in $\mathbb{Z}_p$, $\{(p^n) = p^n\mathbb{Z}_p : n \geq 0\}$ is a FSON of 0.

if N is a neighborhood ~~of~~ with 0 in interior

$\Rightarrow \exists\ B(0,\delta) \subseteq N$

$\quad (p^n)$ s.t. $\frac{1}{p^n} < \delta$.

ex $s \in \mathbb{Z}_p$, consider $\mathcal{F} = \{s + (p^n) = s + p^n\mathbb{Z}_p : n \geq 0\}$

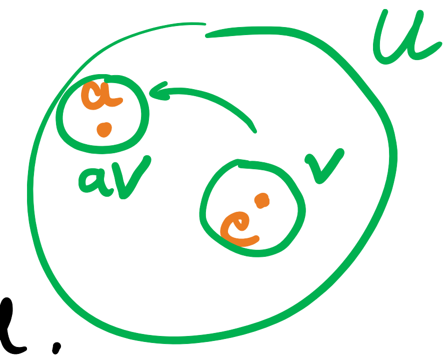
is a FSON of s

ex If G is a top. gp., $\mathcal{F} = \{U_i\}$ is a FSON of e then if $g \in G$

$g\mathcal{F} = \{g * U_i : i \in I\}$ is a FSON of g.

Prop $\mathcal{F}$ is a FSON of $e \in G$. Then:

- (FN1) If $U, V \in \mathcal{F}$, then $\exists\, W \in \mathcal{F}$ s.t. $W \subseteq U \cap V$.
- (FN2) If $a \in U \in \mathcal{F}$, $\exists\, V \in \mathcal{F}$ s.t. $aV \subseteq U$
- (FN3) If $U \in \mathcal{F}$, $\exists\, V \in \mathcal{F}$ s.t. $V^{-1}V \subseteq U$
- (FN4) If $U \in \mathcal{F}$, $x \in G$, $\exists\, V \in \mathcal{F}$ s.t. $x^{-1}Vx \subseteq U$.



Pf (FN4) $f : G \to G$, $a \mapsto x^{-1}ax$ is cont. so $f^{-1}(U)$ open, $e \in f^{-1}(U)$

$\Rightarrow \exists\, V \in \mathcal{F}$, $V \subseteq f^{-1}(U)$ $\Rightarrow$ $x^{-1}Vx \subseteq U$. $\blacksquare$

Note The set $\mathcal{F}$ of all open neigh. of e is FSON of e

w/ properties: • $U, V \in \mathcal{F} \to U \cap V \in \mathcal{F}$, $U^{-1} \in \mathcal{F}$

• $\forall U \in \mathcal{F}$, $\exists W$ symmetric in $\mathcal{F}$ s.t. $W \subseteq U$.

$(W^{-1} = W)$

Prop Let G be a gp., $\mathcal{F} \neq \emptyset$ collection of subsets of G, each containing e, s.t. $\mathcal{F}$ satisfies (FN1) – (FN4)

then $\exists!$ T on G s.t. G is a top gp and $\mathcal{F}$ is a FSON of e.

Cor Any gp G can be made into a top. gp. by specifying as FSON of e a set $\mathcal{F}$ s.t.

• FS1: if $U, V \in \mathcal{F}$, then $\exists W \in \mathcal{F}$ s.t. $W \subseteq U \cap V$

• FS2: if $U \in \mathcal{F}$, $x \in G$, $\exists V \in \mathcal{F}$ s.t. $x^{-1} V x \subseteq U$.

ex $(\mathbb{Q}, \mathcal{T}_p)$

For $t \in \mathbb{Z}$, $U_t = \{ \frac{mp^t}{n} : m,n \in \mathbb{Z}, p \nmid n \} = \{ q \in \mathbb{Q} : \nu_p(q) \geq t \}$

$$\cdots \supseteq U_{-1} \supseteq U_0 \supseteq U_1 \supseteq \cdots \text{ is a FSON of } 0.$$

ex $(\mathbb{Q}^*, \mathcal{T}_p)$ $\mathcal{F} := \{ V_t = 1 + U_t \}_{t \in \mathbb{Z}}$ ~~$t\in$~~ is a FSON of 1.

ex G any gp, $\mathcal{F} = \{$all subgps of finite index$\}$ FSON of e.

ex R comm. ring, $R[[x]]$, $U_n = x^n R[[x]]$

$$U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots \text{ is a FSON of } 0.$$

<u>Note</u> FS2 is trivial if G is abelian.

ex What are the subgrps of finite index in $GL(2,\mathbb{Z}_p)$?

$$GL(2,\mathbb{Z}_p) \xrightarrow{\pi_n} GL(2, \mathbb{Z}/p^n\mathbb{Z})$$

π_n is cont

Let $\underbrace{H_n}_{\text{open}} \overset{\text{subset}}{\underset{\text{subgp}}{\subseteq}} \overbrace{GL(2,\mathbb{Z}/p^n\mathbb{Z})}^{\text{finite, disc. top}}$, $\underset{\text{subgp}}{H} = \pi_n^{-1}(H_n) \to H$ is open

then $[GL(2,\mathbb{Z}_p) : H]$ is finite b/c $\overset{\text{finite.}}{\dfrac{GL(2,\mathbb{Z}_p)}{H}} \longleftrightarrow \overset{\text{finite}}{\dfrac{GL(2,\mathbb{Z}/p^n\mathbb{Z})}{H_n}}$

$$aH \longmapsto \bar{a}H_n$$

- injective: $aH \neq bH \Leftrightarrow ab^{-1} \notin H \Leftrightarrow \pi_n(ab^{-1}) \notin H_n \Leftrightarrow \bar{a}H_n \neq \bar{b}H_n$
- surjective: $\bar{a}H_n \longleftarrow aH$

Now suppose $H \subseteq GL(2,\mathbb{Z}_p)$ is a subgp of finite index, index d.

$\rightarrow H = (H_1, H_2, H_3, \dots) \subseteq GL(2,\mathbb{Z}_p)$

$\quad\quad\quad$ mod p $\quad p^2 \quad p^3$

and $GL(2,\mathbb{Z}/p^n\mathbb{Z})/H_n$ needs to stabilize at some point, to d.

for some $n \geq N$

$\rightarrow H_{N+m} \twoheadrightarrow H_N$ and $(\pi_N^{N+m})^{-1}(H_N) = H_{N+m}$

$\Rightarrow H = \pi_N^{-1}(H_N)$

So <u>all</u> subgps of finite index in $GL(2,\mathbb{Z}_p)$ are inverse images of a subgp at a finite level N.