Math 5020

$G_{\mathbb{Q}}$

Galois Representations

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§. Separation axioms

Def Let X be a top. sp.

(i) X is a T_1-space if $a \neq b$ in X then $\exists$ neigh. of a not containing b. (iff every one-point subset is closed)

$X - \{b\} =$ a union of open sets

(ii) X is a T_2-space (or Hausdorff) if $a \neq b$ in X

$\exists$ neigh U, V, $a \in U$, $b \in V$ st. $U \cap V = \emptyset$.

Prop TFAE:

(i) X is Hausdorff

(ii) $\delta : X \longrightarrow X \times X$, $x \mapsto (x, x)$ is a closed map

(iii) $\forall f, g : Y \to X$ cont. maps, $Z = \{ y : f(y) = g(y) \}$ is closed in Y.

Prop Let G be a top. gp., $\mathcal{F}$ a FSON of e in G

TFAE:
(i) G is Hausdorff
(ii) $f : G \longrightarrow G \times G$ is a closed map
(iii) $\forall f, g : H \to G$ in $\mathcal{T}G$, $\{h : f(h) = g(h)\}$ is a closed subgp of H.

(iv) For any $f : H \to G$ in $\mathcal{T}G$, $\operatorname{Ker} f$ is a closed subgp of H.

(v) $\{e\}$ is a closed subgp of G
(vi) G is T_1, i.e., every one-point subset of G is closed.
(vii) $\bigcap \mathcal{F} = \{e\}$
(viii) The inters. of __all__ neigh. of e is $\{e\}$.

ex $\mathbb{Z}_p$, $\mathcal{F} = \{(p^n) : n \geq 1\}$, $\bigcap \mathcal{F} = \{0\} \overset{(vii)}{\Longrightarrow} \mathbb{Z}_p$ is Hausdorff.

Prod If $\{X_i\}_{i\in I}$ is a family of non-empty spaces. then
$X = \prod X_i$ is Hausdorff $\Leftrightarrow$ all the X_i are Hausdorff.

Prop Let G, G_i be top gps, $H \overset{\text{subgp}}{\subseteq} G$. Then
(i) G Hausdorff $\Rightarrow$ H Hausdorff
(ii) G/H is Hausdorff $\Leftrightarrow$ H is a closed subgp
(iii) H, G/H Hausdorff $\Rightarrow$ G is Hausdorff
(iv) $\prod G_i$ is Hausdorff $\Leftrightarrow$ every G_i is Hausdorff.

ex $\mathbb{Z}_p$ is Hausdorff $\Rightarrow$ all its subgps are Hausdorff

eg G_i are finite $\Rightarrow$ Hausdorff $\Rightarrow$ $\prod G_i$ is Hausdorff $\Rightarrow$ $\{I, G_i, f_j^i\}$
$\varprojlim G_i \subseteq \prod G_i$ also Hausdorff!

§. Open subgroups

Prop Let G be a top. grp. Then:

(i) every open subgrp of G is closed. (open $\Rightarrow$ closed)

(ii) every closed subgrp of finite index is open (closed + fin. index $\Rightarrow$ open)

(iii) every subgrp of G containing a neigh of e is open

(iv) if H is a subgrp of G then G/H is discrete iff H is open.

Pf. (i) H open $\Rightarrow$ gH open $\Rightarrow$ $G - H = \bigcup_{g \notin H} gH$ is open $\Rightarrow$ H is closed.

(ii) $H \subseteq G$ closed of finite index $\Rightarrow$ $G - H = \underbrace{\bigcup_{g \notin H} gH}_{\text{finite \# of cosets}}$ is closed $\Rightarrow$ H is open.

fin. index $\Rightarrow$ finite # of cosets

(iii) $e \in \underset{\text{open}}{\mathcal{U}} \subseteq H$ $\Rightarrow$ $H = \overset{\text{open}}{\overbrace{\mathcal{U}}} \cdot H$ $\left(\begin{array}{l} e \in \mathcal{U} \Rightarrow eH = H \subseteq \mathcal{U}H \\ \mathcal{U} \subseteq H \Rightarrow \mathcal{U}H \subseteq H \end{array} \right)$

and

$= \bigcup_{h \in H} \mathcal{U}h$ open

(iv) G/H is discrete $\Leftrightarrow$ all points $\{gH\}$ are open in G/H

$\Leftrightarrow$ H and left cosets of H are open in G

$\Leftrightarrow$ H is open in G.

ex

$$\mathcal{U} = \{z \equiv 6 \bmod 27\} \subseteq \mathbb{Z}_3$$

$U = B(6, \frac{1}{9})$ is open

$$\mathbb{Z}_3 - \mathcal{U} = \bigcup_{\substack{n=0 \\ n \neq 6}}^{26} \{z \equiv n \bmod 27\} = \bigcup_{\substack{n=0 \\ n \neq 6}}^{26} B(n, \tfrac{1}{9})$$

, $\mathbb{Z}_3 - \mathcal{U}$ is open

$\mathcal{U}$ is open and closed.

§. Connectedness

Def. X is connected if $X \neq \emptyset$ and $X \neq A \cup B$,
$A \cap B = \emptyset$, A, B open in X

$\Longleftrightarrow$ only sets that are open and closed are $\emptyset, X$.

Cor (i) A connected top. gp has no proper open subgps.

(ii) A connected top. gp is generated as an abstract gp.
by any neigh. of e.

Prop (i) If X is connected, $f: X \to Y$ is cont. $\to$ $f(X)$ is connected

(ii) $\prod X_i$ connected $\Leftrightarrow$ every X_i is connected.

Prop Let G, G_i be top. gps, $H \subseteq G$ a subgp. Then

(i) G connected $\Rightarrow$ G/H connected

(ii) H connected and G/H connected $\Rightarrow$ G connected.

(iii) $\prod G_i$ conn. $\Leftrightarrow$ every G_i is conn.

Def A space X is totally disconnected if each connected component of X has just one point.

Cor Any product of discrete spaces is totally disconnected.

Prop Let G, G_i be top. grps., $H \subseteq G$ subgp. Then

(i) If G is tot. disconnected, then so is H.

(ii) H, G/H are tot. disc. $\Rightarrow$ G tot disc.

(iii) $\prod G_i$ is tot disc $\Leftrightarrow$ each G_i is tot. disconnected

ex $\mathbb{Z}_p$ is tot. disconnected

(a) $\mathbb{Z}/p^n\mathbb{Z}$ discrete, finite $\Rightarrow$ tot disconnected

$\Rightarrow \prod \mathbb{Z}/p^n\mathbb{Z}$ is tot. disc.

$\Rightarrow \mathbb{Z}_p \subseteq \prod \mathbb{Z}/p^n\mathbb{Z}$ is tot. disc.

(b) $a, b \in \mathbb{Z}_p$, $a \neq b$. $\Rightarrow \exists\, n$. st. $a \not\equiv b \bmod p^n$

§. Compactness

Def. A space X is compact if it has the Heine-Borel property:

- every open cover of X, $X = \bigcup_{i \in I} X_i$, X_i is open, can be reduced to a finite subcover $\exists\, i(1), \dots, i(n) \in I$ s.t.

$$X = X_{i(1)} \cup \dots \cup X_{i(n)}$$

(equiv. every family of closed subspaces $\{C_i\}$ has the finite inters. property: if each finite sub. of $\{C_i\}$ has non-empty inters then the whole family has non-empty inters.)

- X is seq. compact if it has the Bolzano-Weierstrass prop. (every infinite ~~[illegible]~~ subspace of X has a pt of accumulation $b \in X$)

Fact:
- If (a) X is a metric space or
  (b) X has a countable basis of open sets then
  compact $\Leftrightarrow$ seq. compact.
- A discrete space is compact $\Leftrightarrow$ it is finite

ex $\mathbb{Z}_p$ has prop. (a) and (b).

Prop
(i) Any closed subspace of a compact space is compact.
(ii) Any compact subspace of a Hausdorff space is closed.
(iii) The image of a compact space under a cont. map is compact.
(iv) The union of a finite number of compact subspaces is compact
(v) If C is compact, H Hausdorff, $f: C \to H$ cont $\Rightarrow$ f is a closed map.
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(vi) If $f : C \to H$ as in (v) is a bijection $\Rightarrow$ homeom.

(vii) If $f : C \to H$ as in (v) is a surj $\Rightarrow$ H has quot top

($U \subseteq H$ open $\Leftrightarrow$ $f^{-1}(U)$ open)

ex $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ is compact (quot top)

$\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ cont.

closed bounded

Heine-Borel $\Rightarrow$ compact $\Rightarrow$ image  is compact $\Rightarrow$ $\mathbb{R}^n/\mathbb{Z}^n$ is compact.

Thm (Tychonoff's Theorem)

Any product of compact spaces is compact.

(Converse is also true: $\prod X_i$ compact $\to$ X_i compact)

$\pi_i : \prod X_i \twoheadrightarrow X_i$
cont.

Prop Let $G, G_i \in TG$, $H \leq G$ is a subgp.

(i) G compact + H closed $\Rightarrow$ H is compact

(ii) G compact $\Rightarrow$ G/H is compact

(iii) H compact, G/H compact $\Rightarrow$ G is compact

(iv) $\prod G_i$ compact $\Leftrightarrow$ each G_i is compact.

ex $\mathbb{Z}_p$ is compact.

$$\underbrace{\mathbb{Z}_p}_{\substack{\text{closed subsp.} \\ \Downarrow \\ \text{compact}}} \subseteq \underbrace{\prod_{n=1}^{\infty} \underbrace{\mathbb{Z}/p^n\mathbb{Z}}_{\substack{\text{disc. fin} \\ \Rightarrow \text{compact}}}}_{\text{Tychonoff's} \Rightarrow \text{compact.}} = X$$

Q: $\mathbb{Z}_p$ closed in $\underbrace{\prod \mathbb{Z}/p^n\mathbb{Z}}_{= X}$?

$X - \mathbb{Z}_p$ = non-coherent sequences!

$$a = (a_1, a_2, \ldots, a_n \ldots) \text{ s.t. } \exists n \text{ s.t. } a_n \not\equiv a_{n+1} \bmod p^n$$

$$\in \underbrace{\pi_n^{-1}(\{a_n\})}_{\text{open}} \cap \underbrace{\pi_{n+1}^{-1}(\{a_{n+1}\})}_{\text{open}} = \text{open } \mathcal{U}$$

$a \in \mathcal{U} \subseteq X - \mathbb{Z}_p \Rightarrow a$ interior in $X - \mathbb{Z}_p \Rightarrow X - \mathbb{Z}_p$ is open $\Rightarrow \mathbb{Z}_p$ closed.