MATH 5020

Galois Representations

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§ Separation axioms

Let $X$ be a top. sp.

(i) $X$ is a $T_0$-space if $a \neq b$ in $X$ then $\exists$ neigh. of $a$
not containing $b$. (If every one-point subset is closed)

$X$ has a min of open sets

(ii) $X$ is a $T_1$-space (or Hausdorff) if $a \neq b$ in $X$

$\exists$ neigh $U, V$, a cell, be $V$ st. $U \cap V = \emptyset$.

Proof:

TPAE:

(i) $X$ is Hausdorff

(ii) $f : X \rightarrow X \times X$, $x \rightarrow (x, x)$ is a closed map

(iii) $\forall f, g : Y \rightarrow X$ cont. maps, $Z = \{(x, y) : \exists z \in Y \mid f(y) = g(z) \}

\text{is closed in } X.
Prop Let $G$ be a top. gr., $F$ a Fson of $e \in G$

TFAE: 
(i) $G$ is Hausdorff

(ii) $f: G \to G \times G$ is a closed map

(iii) $f_g, g: H \to G$ in $T_G$, $h: f(g) = g(h)$ is a closed subset of $H$.

(iv) $f_H, g: H \to G$ in $T_G$, $Ker f$ is a closed subset of $H$.

(v) $F$ is a closed subset of $G$

(vi) $G$ is T4, i.e., every one-point subset of $G$ is closed.

(vii) $\bigcap F = \{e\}$

(viii) The inters. of all weight of $e$ is $\{e\}$

$\Rightarrow Z_p, \ F = \{p^n\}_{n \geq 1}, \ \bigcap F = \{0\} \Rightarrow \Rightarrow Z_p \ is \ Hausdorff.$
Proof

If \( \{X_i\}_{i \in I} \) is a family of non-empty spaces, then

\[ X = \prod X_i \text{ is Hausdorff} \iff \text{all the } X_i \text{ are Hausdorff.} \]

Proof

Let \( G, G_i \) be top. gps., \( H \leq G \). Then

(i) \( G \text{ Hausdorff } \implies H \text{ Hausdorff} \)

(ii) \( G/\mathcal{H} \) is Hausdorff \( \iff H \text{ is a closed subgps.} \)

(iii) \( H, G/\mathcal{H} \text{ Hausdorff } \implies G \text{ is Hausdorff} \)

(iv) \( \prod G_i \text{ is Hausdorff } \iff \text{every } G_i \text{ is Hausdorff.} \)

(v) \( \mathbb{Z}_p \text{ is Hausdorff } \implies \text{all its subgps are Hausdorff} \)

(vi) \( G_i \text{ finite } \implies \text{Hausdorff } \Rightarrow \prod G_i \text{ is Hausdorff } \Rightarrow \exists i, G_i, f_i \)

\[ \text{Thus } G_i \leq \prod G_i \text{ also Hausdorff!} \]
6. Open subgroups

Proof. Let \( G \) be a top. grp. Then:

(i) every open subgroup of \( G \) is closed. \( (\text{open} \implies \text{closed}) \)
(ii) every closed subgroup of finite index is open. \( (\text{closed} \implies \text{finite index} \implies \text{open}) \)
(iii) every subgroup of \( G \) containing a neighborhood of \( e \) is open
(iv) if \( H \) is a subgroup of \( G \) then \( G/H \) is discrete \( \iff \) \( H \) is open.

Pf. (i) \( H \) open \( \implies \ gH \) open \( \implies G-H = U gH \) is open \( \implies H \) is closed.
\[ gH \]
(ii) \( H \in G \) closed of finite index \( \implies G-H = U \) \( gH \) is closed \( \implies H \) is open.
\[ gH \]
(iii) \( e \in U \subseteq H \implies H = U \cdot H \) \( (e \in U \implies eH = H \subseteq UH) \)
\[ U \cdot H \] \( \text{open} \)
\[ \text{finite index} \implies \text{finite index} \]
(iv) \( G/H \) is discrete if all points of \( G/H \) are open in \( G/H \)
\( \Rightarrow \) \( H \) is left open in \( G \)
\( \Rightarrow \) \( H \) is open in \( G \).

\[ U = \{ z = 6 \text{ mod } 2 \pi t \leq Z, t = 0 \text{ mod } 2 \pi \} \]
\[ U = B(6, \frac{1}{2}) \cup \prod \]
\[ Z_3 - U = \bigcup_{n=0}^{26} \{ z = n \text{ mod } 2 \pi t = 0 \} \]
\[ B(n, \frac{1}{2}) \cup \prod \]
\[ Z_3 - U \text{ is open} \]
\[ U \text{ is open and closed.} \]
$\S$ Connectedness

Def. $X$ is connected if $X \neq \emptyset$ and $X \neq A \cup B$

$A \cap B = \emptyset$, $A, B$ open in $X$

$\iff$ only sets that are open and closed are $\emptyset, X$.

Cor.

(i) A connected top. $\Rightarrow$ has no proper open subtops.

(ii) A connected top. $\Rightarrow$ is generated $\mathfrak{g}$ as an abstract $\mathfrak{g}$

by any weight of $\mathfrak{e}$.

Prgp.

(i) If $X$ is connected, $f: X \to Y$ is cont. $\Rightarrow f(X)$ is connected.

(ii) $\Pi X_i$ connected $\iff$ every $X_i$ is connected.
Prop Let $G, G_i$ be top. sets. $H \leq G$ a subgroup. Then
(i) $G$ connected $\implies G/H$ connected
(ii) $H$ connected $\iff G/H$ connected $\iff G$ connected.
(iii) $TG$ conn. $\iff$ every $G_i$ is conn.

Def A space $X$ is totally disconnected if each component of $X$ has just one point.

Cor Any product of discrete spaces is totally disconnected.
Prop Let $G, G'$ be top. grps., $H \subseteq G$ subgrp. Then

(i) If $G$ is tot. disconnected, then so is $H$.
(ii) $H, G/H$ are tot. disc. $\Rightarrow G$ tot. disc.
(iii) $\prod G_i$ is tot. disc $\Rightarrow$ each $G_i$ is tot. disconnected.

$\mathbb{Z}_p$ is tot. disconnected

(ii) $\mathbb{Z}/p\mathbb{Z}$ direct, finite $\Rightarrow$ tot. disconnected

$\Rightarrow \prod \mathbb{Z}/p\mathbb{Z}$ is tot. disc.

$\Rightarrow \mathbb{Z}_p \subseteq \prod \mathbb{Z}/p\mathbb{Z}$ is tot. disc.

(iv) $a, b \in \mathbb{Z}_p, a \neq b \Rightarrow$ there st. $a \not\equiv b \mod p^n$
§ 6. Compactness

Def. A space $X$ is compact if it has the Heine-Borel property:

- every open cover of $X$, $X = \bigcup_{i \in I} X_i$, $X_i$ is open, can be reduced to a finite subcover $\{X_{i_1}, \ldots, X_{i_n}\}$ s.t.

  \[ X = X_{i_1} \cup \cdots \cup X_{i_n} \]

(equiv. every family of closed subspaces $\mathcal{C}_f$ has the finite intersection property: if each finite sub. of $\mathcal{C}_f$ has non-empty int., then the whole family has non-empty int.)

- $X$ is sequentially compact if it has the Bolzano-Weierstrass property:

  (every infinite subset of $X$ has a pt. of accumulation in $X$)
**Fact.** If (a) \( X \) is a metric space or
(b) \( X \) has a countable basis of open sets the compact \( \rightarrow \) reg. compact.

- A discrete space is compact \( \iff \) it is finite.

**Prop.** (i) Any closed subspace of a compact space is compact.
(ii) Any compact subspace of a Hausdorff space is closed.
(iii) The image of a compact space under a cont. map is compact.
(iv) The sum of a finite union of compact subspaces is compact.
(v) If \( C \) is compact and Hausdorff, \( f: C \to H \) cont. \( \iff \) \( f \) is a closed map.
(vi) If \( f : C \to H \) as in (v) is a bijection \( \Rightarrow \) homeomorphism.

(vii) If \( f : C \to H \) as in (v) is a surj \( \Rightarrow \) \( H \) has quot top.

\[ U \subseteq H \text{ open} \iff f^{-1}(U) \text{ open} \]

\[ T^n = \mathbb{R}^n / \mathbb{Z}^n \text{ is compact (quot top)} \]

\[ \mathbb{R}^n \to \mathbb{R}^n / \mathbb{Z}^n \text{ cont.} \]

Here-Borel \( \Rightarrow \) compact \( \Rightarrow \) new \( G \) is compact \( \Rightarrow \) \( \mathbb{R}^n / \mathbb{Z}^n \) is compact.
Then (Tychonoff’s Theorem)
Any product of compact spaces is compact.
(Converse is also true: \( \prod X_i \text{ compact} \rightarrow X_i \text{ compact} \))
\[
\pi_i : \prod X_i \rightarrow X_i
\]
\text{cont.}

Proof
Let \( G, G_i \in \mathcal{T}_G \), \( H \leq G \) is a subgp.
(i) \( G \) compact \( \Rightarrow \) \( H \) closed \( \Rightarrow \) \( H \) is compact
(ii) \( G \) compact \( \Rightarrow \) \( G/H \) is compact
(iii) \( H \) compact, \( G/H \) compact \( \Rightarrow \) \( G \) is compact
(iv) \( \prod G_i \) compact \( \Rightarrow \) each \( G_i \) is compact.
$\mathbb{Z}_p$ is compact.

$\mathbb{Z}_p \subseteq \prod_{n=1}^{\infty} \mathbb{Z}/p^n \mathbb{Z}$

Closed, hence compact.

Compact $\implies$ Tychonoff's $\implies$ compact.

Q: $\mathbb{Z}_p$ closed in $\prod \mathbb{Z}/p^n \mathbb{Z}$?

$X - \mathbb{Z}_p = \text{non-coherent sequences!}$

$a = (a_1, a_2, \ldots, a_n, \ldots)$ s.t. $\forall n \exists i \text{ s.t. } a_n \neq a_{n+i}$ and $p^n \nmid a_{n+i} - a_n$ $\in \prod_{n=1}^{\infty} (\mathbb{Z}/p^n \mathbb{Z}) = \bigcap_{n=1}^{\infty} \prod_{m=1}^{n} (\mathbb{Z}/p^m \mathbb{Z}) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{n} \mathbb{Z}/p^m \mathbb{Z}$ open $U = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{n} \mathbb{Z}/p^m \mathbb{Z}$ open

$a \in U \subseteq X - \mathbb{Z}_p$ so $a$ stably in $X - \mathbb{Z}_p$ $\implies X - \mathbb{Z}_p$ is open $\implies \mathbb{Z}_p$ closed.