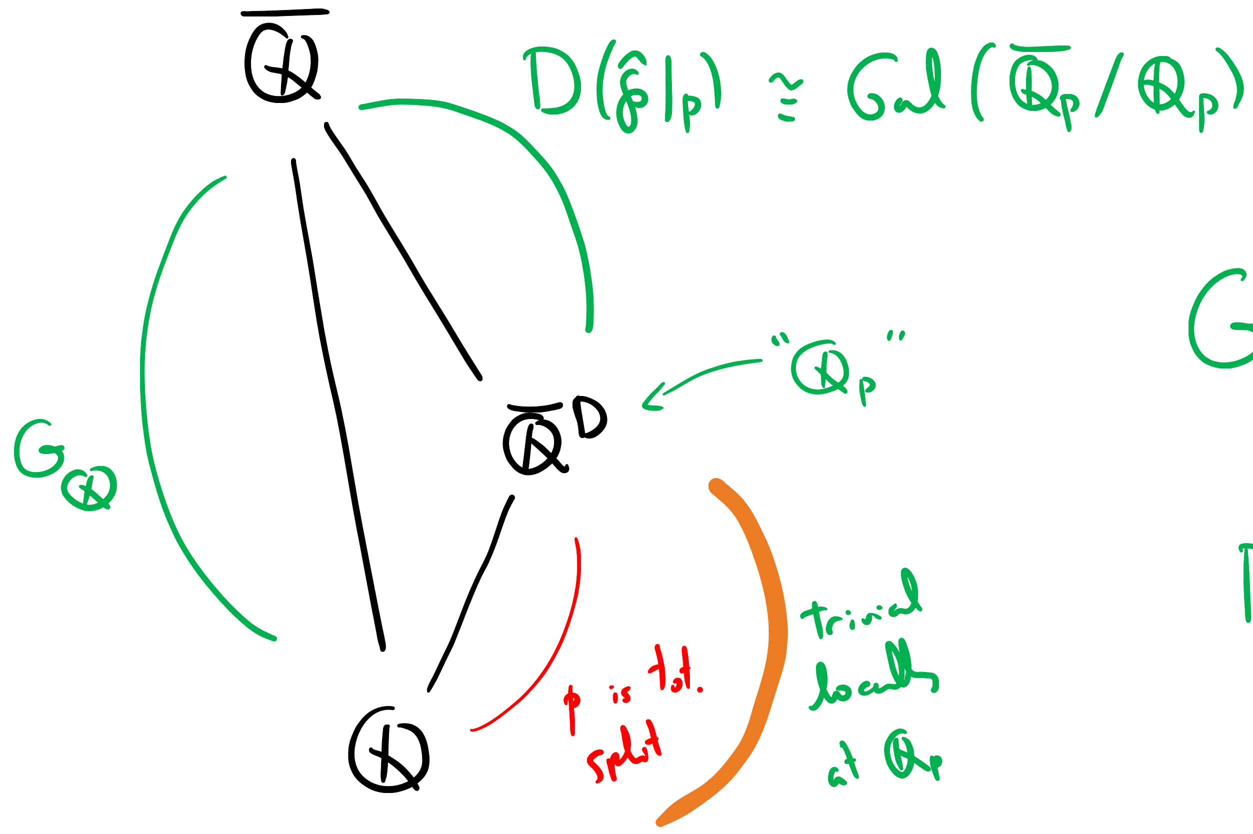
Galois Representations : $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

$$
\begin{tikzcd}
\overline{\mathbb{Q}} \arrow[dd, dash] \arrow[dd, bend right, dash, "G_{\mathbb{Q}}"'] \arrow[dr, dash] & \\
& K \arrow[dl, dash] \arrow[dl, bend left, dash, "G"] \\
\mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{Z}} \arrow[d, dash] \\
\mathcal{O}_K \arrow[d, dash] \\
\mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
p\,\mathcal{O}_K = (\mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_r)^e \arrow[d, dash] \\
p
\end{tikzcd}
\qquad
\begin{tikzcd}
\hat{\mathfrak{p}} \arrow[d, dash] \\
\mathfrak{p}_1 \arrow[d, dash] \\
p
\end{tikzcd}
$$

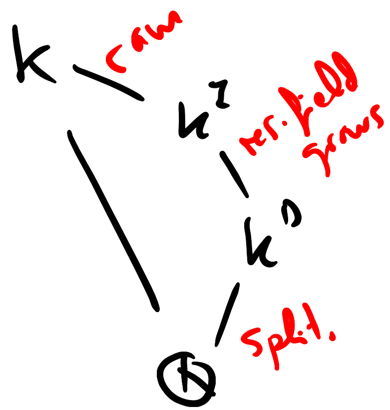
$$\mathrm{Gal}(K/\mathbb{Q}) \supseteq D(\mathfrak{p}_1|p) \supseteq I(\mathfrak{p}_1|p)$$

$$D(\hat{\mathfrak{p}}|p) = \varprojlim_{K/\mathbb{Q}} D(K/\mathbb{Q}, \mathfrak{p}|p)$$

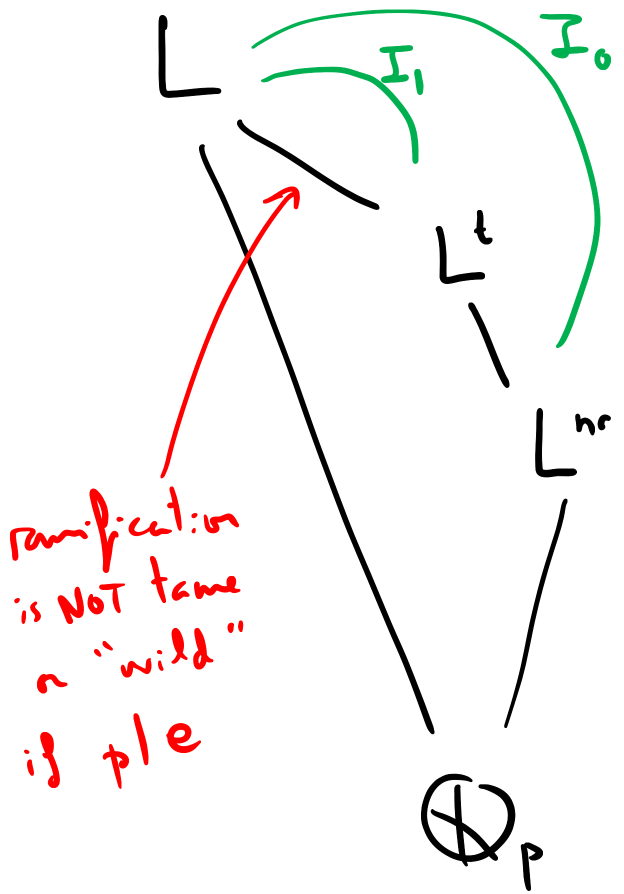
$$\mathfrak{p} = \hat{\mathfrak{p}} \cap \mathcal{O}_K$$



$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \supseteq D(\hat{\mathfrak{p}}|p) \cong \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$$



$\overline{\mathbb{Q}}_p / \mathbb{Q}_p$?



Start w/ $L/\mathbb{Q}_p$ finite Galois ext'n.

$\mathrm{Gal}(L/\mathbb{Q}_p) = D(p) \supseteq I(p) = I_0$

L^{nr} = largest ext'n in L s.t. $L^{nr}/\mathbb{Q}_p$ is unramified.

$p\mathcal{O}_{L^{nr}} = (\pi)^e \quad \rightarrow \quad e = 1$

L^t = largest ext'n of $\mathbb{Q}_p$ in L s.t. the ramification is "tame"

$p\mathcal{O}_{L^t} = (\pi)^e \qquad \gcd(e,p) = 1.$

Fixed by "wild inertia" I_1 = p-primary comp. of I_0

$$\begin{tikzcd}
\overline{\mathbb{Q}}_p \arrow[d, dash] \arrow[d, phantom, bend left=60, "I_1"] \arrow[ddd, phantom, bend left=80, "I_0"] \\
\mathbb{Q}_p^t \arrow[d, dash] \arrow[d, phantom, bend right=60, "\prod_{\ell\neq p}\mathbb{Z}_\ell"'] \\
\mathbb{Q}_p^{nr} \arrow[d, dash] \arrow[d, phantom, bend left=60, "\hat{\mathbb{Z}}"] \\
\mathbb{Q}_p
\end{tikzcd}$$

$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \supseteq I_0(p)$

$\mathbb{Q}_p^{nr}$ = max'l unramified ext'n of $\mathbb{Q}_p$

$K/\mathbb{Q}_p$ Gal finite unramified degree n $\longleftrightarrow$ $k/\mathbb{F}_p$ degree n (unique!)

$$\mathrm{Gal}(\mathbb{Q}_p^{nr}/\mathbb{Q}_p) \cong \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}} = \langle 1 \rangle$$

top cyclic.

NOTE: $\mathbb{Q}_p^{nr}$ contains $\mathbb{Q}_p(\zeta_n)$ for any $(n,p)=1$.

$\mathbb{Q}_p^t$ = max'l tamely ramified ext'n of $\mathbb{Q}_p$

$\mathbb{Q}_p^t/\mathbb{Q}_p^{nr}$ $\quad$ $K/\mathbb{Q}_p^{nr}$, $K \subseteq \mathbb{Q}_p^t$ $\overset{!}{\Longrightarrow}$ $K = \mathbb{Q}_p^{nr}(\pi^{1/d})$ w/ $(p,d)=1$

$$\mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p^{nr}) \cong \varprojlim_{\substack{k/\mathbb{Q}_p^{nr} \\ k \subseteq \mathbb{Q}_p^t}} \mathrm{Gal}(k/\mathbb{Q}_p^{nr}) \cong \varprojlim_{(d,p)=1} \mathbb{Z}/d\mathbb{Z}$$

$$\overset{??}{\cong} \prod_{\ell \neq p} \mathbb{Z}_\ell = \langle 1 \rangle$$

top cyclic.

$$\begin{tikzcd}
\overline{\mathbb{Q}}_p \arrow[d, dash] \arrow[ddd, dash, bend right] \\
\mathbb{Q}_p^t \arrow[d, dash, "\cong \prod_{\ell \neq p} \mathbb{Z}_\ell"] \\
\mathbb{Q}_p^{nr} \arrow[d, dash, "\cong \hat{\mathbb{Z}}"] \\
\mathbb{Q}_p
\end{tikzcd}$$

$$\mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p) = \overline{\langle \sigma, \tau \rangle}$$

s.t. $\overline{\langle \sigma \rangle} \cong \hat{\mathbb{Z}}$, $\overline{\langle \tau \rangle} \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$, $\sigma^{-1}\tau\sigma = \tau^{p}$

$\#\mathbb{F}_p$

$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^t)$ is a pro-p gp generated by <u>two</u> elements.

Reference??

Galois Representations: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

REF: Serre's "Abelian ℓ-adic Gal. repn's"

K field

K^{sep} sep. alg. closure of K

$G_K = \mathrm{Gal}(K^{sep}/K)$ w/ Krull topology (compact, tot. disc.)

ℓ (ell) a prime number, $\mathbb{Q}_\ell$ ℓ-adics

V finite dim'l vector space over $\mathbb{Q}_\ell$

a gp, a manifold (ℓ-adic analytic manifold)

$\mathrm{Aut}(V)$ automorphisms of $V/\mathbb{Q}_\ell$, is an $\underbrace{\ell\text{-adic Lie gp}}$

If $\dim V = n$, then $\mathrm{Aut}(V) \cong GL(n, \mathbb{Q}_\ell)$ pick a basis.

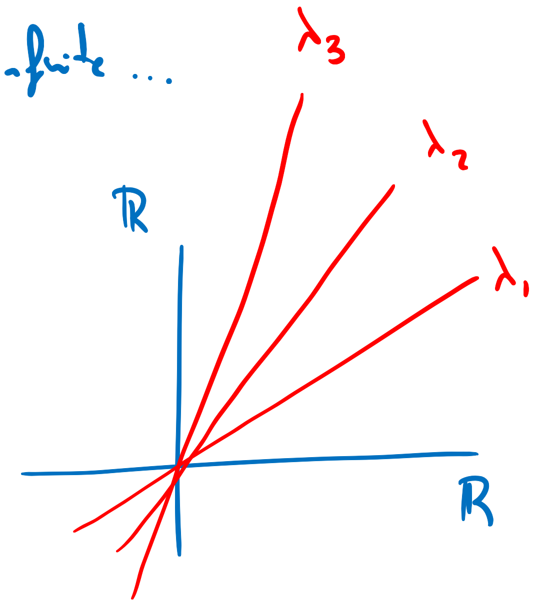
Def An ℓ-adic Gal. repn's, or an ℓ-adic rep'n of G (or of K) is a "continuous" homomorphism $\rho : G \longrightarrow \mathrm{Aut}(V) \cong GL(n, \mathbb{Q}_\ell)$

topology?? $GL(n, \mathbb{Z}_\ell)$?

Topology on $\mathrm{Aut}(V)$

- Discrete top. ... $\mathrm{Aut}(V)$ is infinite ...
- $\mathrm{Aut}(V) \cong GL(n, \mathbb{Q}_\ell)$
- $V = \mathbb{R}$, over $\mathbb{R}$ $\quad \mathrm{Aut}(\mathbb{R})$

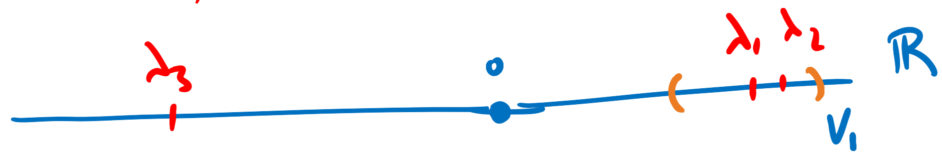
$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto \lambda x$$



$GL(n, \mathbb{R}) = \mathbb{R}^{n^2}$

$\cong GL(1, \mathbb{R}) \cong \mathbb{R}^\times$

equiv. topologies.

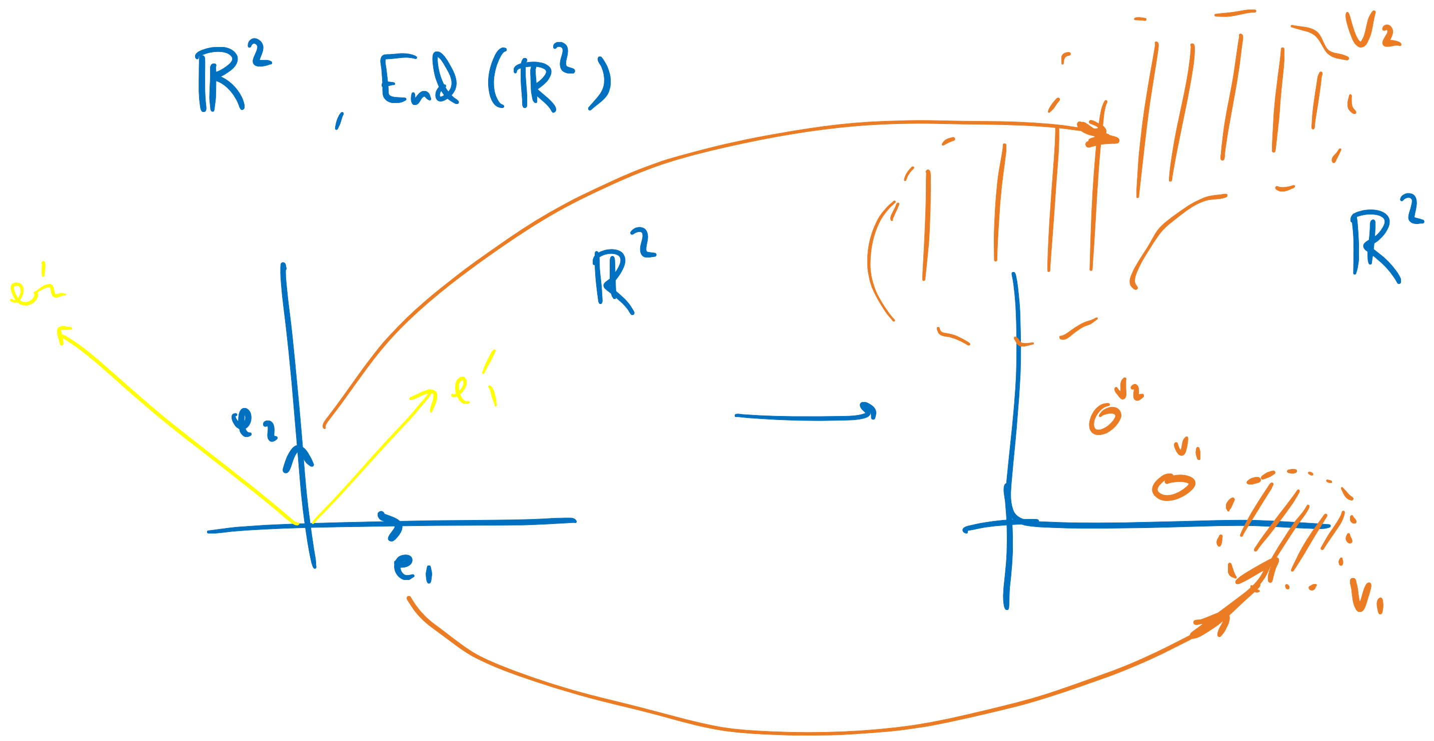


- V finite dim'l $/ F$ field w/ top, $V = \underbrace{\langle w_1, \dots, w_n \rangle}_{\text{basis}}$ $\quad$ (F top $\rightsquigarrow$ V top) $\quad V \cong F^n$

$$\mathrm{Aut}(V) \supseteq \mathcal{U} \text{ open} \iff \mathcal{U} = \{ \phi \in \mathrm{Aut}(V) : \phi(w_i) \subseteq V_i \}$$

where $\underset{\text{open}}{V_i \subseteq V}$

$\mathbb{R}^2$, $\mathrm{End}(\mathbb{R}^2)$



$$\mathcal{U} = \{ \phi : \mathbb{R}^2 \to \mathbb{R}^2 \ : \ \phi(e_1) \subseteq V_1 \ , \ \phi(e_2) \subseteq V_2 \}$$

open of $\mathrm{End}(\mathbb{R}^2)$

$$\phi : \begin{array}{c} \begin{array}{cc} \phi(e_1) & \phi(e_2) \end{array} \\ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \end{array}$$

$\mathcal{U}$ correspond to an open in $GL(2,\mathbb{R})$ where the top is that of $\mathbb{R}^4$ coord. wise.